Question

The function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

is *not* continuous at 0. Give an example of an open set $V \subset \mathbf{R}$ for which $F^{-1}(V)$ is not open, and a closed set $C \subset \mathbf{R}$ for which $F^{-1}(C)$ is not closed. Do likewise for the functions g, h :

$$g(x) = \begin{cases} 2x+3, & x \geq 1 \\ -x+4, & x < 1 \end{cases} \qquad h(x) = \begin{cases} \sin(\frac{1}{x}), & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Answer

Let

$V = (0, \infty)$**; then** $f^{-1}(V) = [0, \infty)$**: not open**

$C = (-\infty, 0]$**; then** $f^{-1}(C) = (-\infty, 0)$**: not closed.**

Let

$V = (4, \infty)$**; then** $g^{-1}(V) = (-\infty, 0) \cup [1, \infty)$**: not open**

$C = (-\infty, 4]$**; then** $g^{-1}(C) = [0, 1)$**: not closed**

Let $V = (4, \infty)$**; then** $g^{-1}(V) = (-\infty, 1) \cup (1, \infty)$**, i.e.** $\Re \backslash \{1\}$**. Then** $h^{-1}(V)$ **consists of the positive** x**-axis with all points** $\dfrac{1}{2n\pi + \frac{\pi}{2}}$**,** $n \in \mathbf{N}$**, removed, together with the negative** x**-axis including** 0**. Thus there is no interval** $(-\epsilon, \epsilon)$ $(\epsilon > 0)$ **entirely contained in** $h^{-1}(V)$**, but** $0 \in h^{-1}(V)$**.**

For C **take the complement of** V**, namely** $\{1\}$**. Then all points** $a_n = (2n\pi + \frac{\pi}{2})^{-1}$ **lie in** C**, and** $a_n \to 0$ **as** $n \to \infty$**, but** $0 \notin C$**.**