REAL ANALYSIS
INTERCHANGE THEOREMS

Theorem 1 Suppose $f_{12}(xy)$ is continuous at (ab) and $\exists\delta > 0$ such that $f_2(xb)$ exists for $|x-a| < \delta$. Then $f_{21}(ab)$ exists and $= f_{12}(ab)$.

Proof Without loss of generality we may take $(ab) = (0,0)$.

Let ε be given. Choose δ, such that $0 < \delta_1 < \delta$ and $|f_{12}(xy) - f_{12}(00)| < \varepsilon$ whenever $|x| < \delta_1$ and $|y| < \delta_1$. Suppose $0 < |h| < \delta_1$.

Consider

$$\frac{f_2(ho) - f_2(00)}{h} = \lim_{k\to 0} \frac{\Delta_{hk}}{P} hk$$

$$\Delta_{hk} = \{f(hk) - f(h0)\} - \{f(ok) - f(00)\}$$

We regard k as being temporarily fixed with $|k|$ sufficiently small, and write $F(h) = f(hk) - f(h0)$ so that

$$\begin{aligned}
\frac{\Delta_{hk}}{hk} &= \frac{F(h) - F(0)}{hk} \\
&= \frac{F'(\theta h)}{k} \text{by MVT} 0 < \theta < 1 \\
&= \frac{f_1(\theta h, k) - f_1(\theta h, 0)}{k} \\
&= f_{12}(\theta h \theta' k) \text{ my MVT} 0 < \theta' < 1
\end{aligned}$$

Hence

$$\left|\frac{\Delta_{hk}}{hk} - f_{12}(00)\right| < \varepsilon.$$

Letting $k \to 0$ we have by (1)

$$\left|\frac{f_2(h0) - f_2(00)}{h} - f_{12}(00)\right| \le \varepsilon.$$

Hence $f_{21}(00)$ exists and is equal to $f_{12}(00)$

In the following results R de3notes the closed rectangle $a \le x \le b$ $c \le y \le d$.

Lemma Let $f(xy)$ be continuous on R. Then we have $\phi(x) = \sum_c^d f(xy)\,dy$ is continuous on $[ab]$.

Proof $f(xy)$ is uniformly continuous on R. Hence, given $\varepsilon > 0$, $\exists \delta > 0 | |f(P) - f(Q)| < \frac{\varepsilon}{d-c}$ whenever $P \in R$ $Q \in R$ and $|PQ| < \delta$.

Now if x_1, x_2 are each in $[ab]$ and $|x_1 - x_2| < \delta$:

$$|\phi(x_1) - \phi(x_2)| \leq \int_c^d |f(x_1 y) - f(x_2 y)|\,dy < d - c\frac{\varepsilon}{d-c} = \varepsilon$$

Theorem 2 Let $f(xy)$ be continuous as a function of y for $c \leq y \leq d$ relative to this interval, for each x with $a \leq x \leq b$. Suppose that $f_1(xy)$ is continuous with respect to (xy) on the region $a < x < b$ $c \leq y \leq d$. Then

$$\frac{\partial}{\partial x}\int_c^d f(xy)\,dy = \int_c^d \frac{\partial}{\partial x} f(xy)\,dy \text{ for } a < x < b$$

.

Proof Let x_0 satisfy $a < x_0 < b$.

Choose η to satisfy $a < a + \eta < x_0 < b - \eta < b$. Let R_η be the closed rectangle $a + \eta \leq x \leq b - \eta, c \leq y \leq d$, and I_η the interval $a + \eta \leq x \leq b - \eta$.

$f_1(xy)$ is uniformly continuous on R_η. Given $\varepsilon > 0 \exists \delta | |f_1(P) - f_1(Q)| < \frac{\varepsilon}{d-c}$ whenever $|PQ| < \delta$ and $P, Q \in R_\eta$.

Then if $|H| < \delta$ and $x_0 + h \in I_\eta$ we have

$$\begin{aligned}
& \left|\frac{\phi(x_0 + h) - \phi(x_0)}{h} - \int_c^d f_1(x_0 y)\,dy\right| \\
= & \left|\int_c^d \left\{\frac{f(x_0 + h, y) - f(x_0 y)}{h} - f_1(x_0 y)\right\} dy\right| \\
= & \left|\int_c^d \{f_1(x_0 + \theta y h, y) - f_1(x_0 y)\}\,dy\right| \quad 0 < \theta_y < 1 \\
< & \; d - c.\frac{\varepsilon}{d-c} = \varepsilon
\end{aligned}$$

Theorem 3 Let $f(xy)$ be continuous on R. Then

$$I_1 = \int_a^b dx \int_c^a f(xy)\,dy = \int_c^d dy \int_a^b f(xy)\,dx = I_2$$

Note This can be justified under more general conditions.

Proof Subdivide R into n^2 small rectangles. If R_{ij} is one of these then $A = A_{ij}n^2$.

Let $M_{ij} = \overline{bd}_{P\in R_{ij}} f(P)$ $m_{ij} = \underline{bd}_{p\in R_{ij}} f(P)$ $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$.

Since f is uniformly continuous on R we may choose n sufficiently large so that $M_{ij} - m_{ij} < \varepsilon$. Thus for each ij,

$$
\begin{aligned}
m_{ij}\frac{A}{n^2} &\leq \int_{a_{i-1}}^{a_i} dx \int_{b_{j-1}}^{b_j} f(xy)\, dy \leq M_{ij}\frac{A}{n^2} \\
\int_a^b dx \int_c^d dy &= \sum_{i=1}^n \sum_{j=1}^n \int_{a_{i-1}}^{a_i} dx \int_{b_{j-1}}^{b_j} f(xy)\, dy \text{ therefore} \\
\sum_{i=1}^n \sum_{j=1}^n m_{ij}\frac{A}{n^2} \leq \quad I_1 &\leq \sum_{i=1}^n \sum_{j=1}^n M_{ij}\frac{A}{n^2} \text{ similarly} \\
\sum_{i=1}^n \sum_{j=1}^n m_{ij}\frac{A}{n^2} \leq \quad I_2 &\leq \sum_{i=1}^n \sum_{j=1}^n M_{ij}\frac{A}{n^2} \\
&\Rightarrow |I_1 - I_2| < A\varepsilon \Rightarrow I_1 = I_2
\end{aligned}
$$

Theorem 4 Let Tb be the triangular region $a \leq y \leq x \leq b$.

Let $f(xy)$ be continuous on T. Then

$$\int_a^b dx \int_a^x f(xy)\, dy = \int_a^b dy \int_y^b f(xy)\, dx.$$

Proof We first establish the existence of these integrals.

Let $M = \overline{bd}_{P\in T}|f(P)|$. Given $\varepsilon >)$ choose δ such that

1. $M\delta < \frac{1}{2}\varepsilon$
2. $|f(P) - f(Q)| < \frac{1}{2}\frac{\varepsilon}{b-a}$ whenever $|PQ| < \delta$ and $P, Q \in T$.

If $a \leq x_1 \leq x_2 \leq b$ and $x_2 - x_1 < \delta$ we have

$$
\begin{aligned}
& \left|\int_a^{x_2} f(x_2y)\, dy - \int_a^{x_1} f(x_1y)\, dy\right| \\
\leq \quad & \left|\int_a^{x_1} f(x_2y)\, dy - f(x_1y)\, dy + \int_{x_1}^{x_2} f(x_2y)\, dy\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (x_2 - x_1)M + \int_a^{x_1} |f(x_2y) - f(x_1y)|\, dy \\
&\leq \delta M + b - a\frac{1}{2}\frac{\varepsilon}{b-a} = \varepsilon
\end{aligned}
$$

This proves continuity of $\int_a^x f(xy)\, dy$, and hence the existence of the LHS.

Similarly $\int_y^b f(xy)\, dx$ is continuous and hence the RHS exists.

We write $\iint_T f(xy)\, dxdy$ for $\int_a^b\, dx \int_a^x f(xy)\, dx$.

We write $\iint_T f(xy)\, dydx$ for $\int_a^b\, dy \int_y^b f(xy)\, dx$.

Also let $A(T)$ denote the area of T.

We have

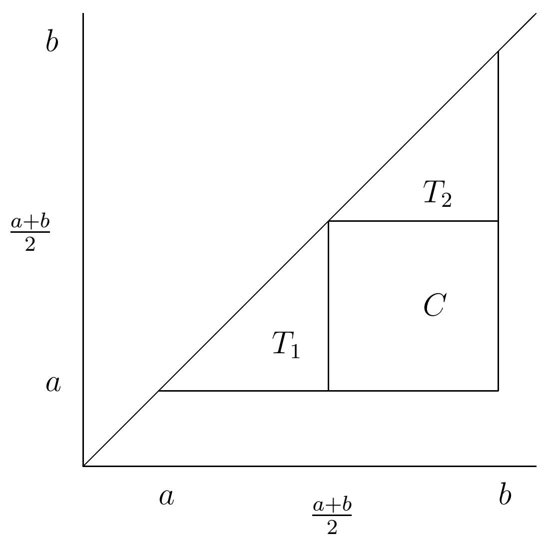
$$
\left| \iint_T f(xy)\, dxdy - \iint_T f(xy)\, dydx \right| \leq 2NA(T) \tag{1}
$$

Population P_r. Let T be the region $a \leq y \leq x \leq b$. Let $f(xy)$ be continuous on T. Let $M = \overline{bd}_{p\in T}|f(P)|$ and let $A(T)$ be the area of T.

Then for $r = 0, 1, 2, \ldots, n$ we have

$$
\left| \iint_T f(xy)\, dxdy - \iint_T f(xy)\, dydx \right| \leq 2MA(T) \left(\frac{1}{2}\right)^r
$$

P_o is true from (1). Let $r \geq 0$ and suppose P_r is true. It remains to prove that P_{r+1} is true.



$$\left|\iint_T f\,dxdy - \iint_T f\,dydx\right| =$$

$$\left|\iint_{TR_1} f\,dxdy - \iint_{T_1} f\,dydx + \iint_{T_2} f\,dxdy - \iint_{T_2} f\,dydx + \iint_C f\,dxdy - \iint_C f\,dydx\right|$$

By Theorem 3 $\iint_C f\,dxdy = \iint_C f\,dydx$ therefore

$$\begin{aligned}
\left|\iint_T f\,dxdy - \iint_T f\,dydx\right| &\leq \left|\iint_{T_1} f\,dxdy - \iint_{T_1} f\,dydx\right| + \left|\iint_{T_2} f\,dxdy - \iint_{T_2} f\,dydx\right| \\
&\leq 2\left(\frac{1}{2}\right)^r M(A(T_1) + A(T_2)) \\
&= 2\left(\frac{1}{2}\right)^r M\left(2.\frac{1}{4}A(T)\right) \\
&= 2\left(\frac{1}{2}\right)^{r+1} MA(T)
\end{aligned}$$

Hence the theorem follows.

The following two theorems are generalisations.

Theorem 5 Suppose $c(x)$ $d(x)$ are continuous on $I = [ab]$ and that $c(x) \leq d(x)$, for x in I. Let S be the region of points (x, y) satisfying $a \leq x \leq b$ $c(x) \leq y \leq d(x)$. Suppose that $f(xy)$ is continuous on S.

Then $\int_{c(x)}^{d(x)} f(xy)\,dy$ is continuous on I.

Theorem 6 Let $c(x)$ $d(x)$ be continuous on $I = [ab]$ Suppose $c(x) < d(x)$ and that $c(x)$ $d(x)$ are differentiable for $a < x < b$. Let S be the set of points with $a \leq x \leq b$ $c(x) \leq y \leq d(x)$. Let S_0 be the set of points with $a < x < b$ $c(x) < y < d(x)$. Suppose that $f(xy)$ is continuous on S and that $f_1(xy)$ is uniformly continuous in S_0. Then we have

$$\frac{d}{dx}\int_{c(x)}^{d(x)} f(xy)\,dy = d'(x)f(x, d(x)) - c'(x)f(x, c(x)) + \int_{c(x)}^{d(x)} f_1(x, y)\,dy$$