REAL ANALYSIS
INEQUALITIES

The Inequality of Arithmetic and Geometric Means Given $x_1 \geq 0$ $x_2 \geq 0 \ldots x_n \geq 0$

$$A = \frac{x_1 + x_2 + \ldots + x_n}{n} \; G = (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$$

Then $G < A$ unless $x_1 = x_2 = \ldots = x_n$, when $G = A$.

Proof Suppose without loss of generality that x_1 is a maximal x_ν and x_2 is a minimal x_ν.

If $x_1 = x_2$ the $_r$'s are all equal and there is nothing to prove.

Suppose then that $x_1 > x_2$. We form a new set of numbers $x_{11} x_{21} \ldots x_{n1}$ by writing

$$x_{11} = A \; x_{21} = -A + x_1 + x_2 \; x_{r1} = x_r \; r = 3, \ldots, n.$$

Let A_1, G_1 be the A.M and G.M of the x_{r1}'s.

$A_1 = A$ since $x_{11} + x_{21} = x_1 + x_2$.

However

$$\begin{aligned} x_{11} x_{2x} - x_1 x_2 &= A(x_1 + x_2 - A) - x_1 x_2 \\ &= (x_1 - A)(A - x_2) > 0 \end{aligned}$$

since $x_1 > A > x_2$.

Therefore $G_1 > G$

If the $x_{\nu 1}$ are not all equal we can again take a largest $x_{\alpha 1}$ and a smallest $x_{\beta 1}$ and replace them by $A, \; x_{\alpha 1} + x_{\alpha 2} - A$.

$A_2 = A_1$ $G_2 > G_1$

After at most $n - 1$ steps all the x's are equal.

$G < G_1 < \ldots < G_K = A_k = A$ therefore $G < A$.

Cauchy's Inequalities Given $x_1 \ldots x_n$ $y_1 \ldots y_n$ real.

Then

$$\sum_{r=1}^{n} x_r y_r \leq \left(\sum_{r=1}^{n} a_r^2\right)^{\frac{1}{2}} \left(\sum_{r=1}^{n} y_r^2\right)^{\frac{1}{2}}$$

with equality $\Leftrightarrow$ the two sets are proportionali.e. $\Leftrightarrow \exists(\lambda\mu) \neq (00)|\lambda x_r + \mu y_r = 0 \ (r = 1, 2, \ldots, n)$

Proof Consider the quadratic form $Q(\lambda\mu)$ defined by

$$\begin{aligned} Q(\lambda\mu) &= \sum_{r=1}^{n}(\lambda x_r + \mu y_r)^2 \\ &= \lambda^2 \sum_{r=1}^{n} x_r^2 + 2\lambda\mu \left(\sum)r = 1^n x_r y_r\right) + \mu^2 \sum_{r=1}^{n} y_r^2 \end{aligned}$$

If $\exists\lambda\mu \neq 00|\lambda x_r + \mu y_r = 0 \ r = 1, 2, \ldots, n$ then there is nothing to prove.

Suppose $\exists$ no such (λ, μ). Then $Q(\lambda\mu) > 0$ for every $(\lambda\mu) \neq (00)$. Hence $Q(\lambda\mu)$ is positive definite so that

$$\left(\sum_{r=1}^{n} x_r y_r\right)^2 < \sum_{r=1}^{n} x_r^2 \sum_{r=1}^{n} y_r^2$$

using "$b^2 < 4ac$".

Weighted Means Given a set of non-negative numbers $x_1 \ldots x_n$ and a set of weights P, where we attach the weight P_r to x_r, each $P > 0$. The weighted means are

$$A_P = \frac{P_1 x_1 + \ldots + P_n x_n}{P_1 + P_2 + \ldots + P_n}$$

$$G_P = (x_1^{P_1} x_2^{P_2} \ldots x_n^{P_n})^{\frac{1}{p_1 + P_2 + \ldots + P_n}}$$

Note If the weights $p_1 \ldots p_n$ are replaced by $tp_1 \ldots tp_n$, then A_p G_P are unchanged. In particular if we take $t = \frac{1}{P_1 + \ldots + p_n}$ we get a set of weights $Q: \ q_1 \ldots q_n | q_1 + \ldots + q_n = 1$. Then $G_P \leq A_P$ with equality $\Leftrightarrow$ all the x's are equal.

Proof (i) Result proved when P_j are all integers.

(ii) Result follows when P_j are commensurable; i.e. when $\exists t > 0 | tP_1 \ldots tP_n$ are all integers.

(iii) We have to deal with the case where the P's are not commensurable.

Let $q_1 \ldots q_n$ be a set of weights $| \sum q_j = 1$.

Let Q $(q_1 \ldots q_n)$ be a point in R_n.

Take a set of rational points

$$P^r = (r_1 \ldots r_n) \quad r_j > 0$$

where $P^r \to Q$ as $r \to \infty$.

$G_{P_r} < A_{P_r}$ unless the x^ν equal.

Letting $r \to \infty$ $G_Q \leq A_Q$.

We still have to prove strict inequality when the x's are not all equal. Suppose then that the x's are not all equal. Write

$$q_j = j'_j + q''_j \; j = 1, 2, \ldots, n$$

where $q'_j > 0$ $q''_j > 0$ q'_j is rational.

$$\begin{aligned} P' &: Q'_1 \ldots q'_n \;\; P'' : q''_1 \ldots q''_n \\ r' &= \sum q'_j \; r'' = \sum q''_j \; r' + r'' = 1 \end{aligned}$$

$G_{P'} < A_{P'}$ by (ii) $G_{p''} \leq A_{p''}$

$$G_Q = (G_{P'})^{r'} (G_{p''})^{r''} \leq r' G_{P'} + r'' G_{P''} < r' A_{P'} + r'' A_{P''} = A_Q$$

using $\sum q_j = 1$

Hölder's Inequality We have two sets of numbers

$$\begin{array}{ll} x_1 \ldots x_n & x_j \geq 0 \\ y_1 \ldots y_n & y_j \geq 0 \end{array}$$

α, β are positive and $\alpha + \beta = 1$. Then

$$\sum_{\nu=1}^{n} x_\nu^\alpha y_\nu^\beta \leq \left(\sum_{\nu=1}^{n} x_\nu \right)^\alpha \left(\sum_{\nu=1}^{n} y_\nu \right)^\beta$$

with equality $\Leftrightarrow$ the sets are proportional.

Alternative Form Suppose λ, μ are positive and $\frac{1}{\lambda} + \frac{1}{\mu} = 1$

$$\sum_{\nu=1}^{n} x_\nu y_\nu \leq \left(\sum_{\nu=1}^{n} x_\nu^\lambda\right)^{\frac{1}{\lambda}} \left(\sum_{\nu=1}^{n} y_\nu^\mu\right)^{\frac{1}{\mu}}$$

[This result above with α, β replaced by $\frac{1}{\lambda}, \frac{1}{\mu}$ and $x_\nu^\alpha, x_\nu^\beta$ replaced by new variables x_ν, y_ν.]

This generalises to k sets and k numbers $\alpha_1 + \ldots + \alpha_k = 1$.

Cauchy's inequality follows with $\lambda = \mu = 2$.

Proof Write $U = \sum_{\nu=1}^{n} x_\nu \;\; V = \sum_{\nu=1}^{n} y_\nu$

Suppose $UV > 0$ (nothing to prove otherwise).

$$\begin{aligned} U^\alpha V^\beta = \sum_{\nu=1}^{n} x_\nu^\alpha y_\nu^\beta &= \sum_{\nu=1}^{n} \left(\frac{x_\nu}{U}\right)^\alpha \left(\frac{y_\nu}{V}\right)^\beta \\ &\leq \sum_{\nu=1}^{n} \alpha \frac{x_\nu}{\nu} + \beta \frac{y_\nu}{\nu} = \alpha + \beta = 1 \end{aligned}$$

with equality $\Leftrightarrow \frac{x_\nu}{U} = \frac{y_\nu}{V}$ for $\nu = 1, 2, \ldots, n$.

These inequalities generalise to integrals.

Suppose $f(x) \geq 0$ $g(x) \geq 0$ are continuous on $[a \; b]$

$$\int_a^b f(x) g(x)\, dx \leq \left(\int_a^b f^2\, dx\right)^{\frac{1}{2}} \left(\int_a^b g^2\, dx\right)^{\frac{1}{2}}$$

This is known as Schwarz's inequality.

If $\frac{1}{\lambda} + \frac{1}{\mu} = 1$ $\lambda > 0$ $\mu > 0$ then

$$\int_a^b f(x) g(x)\, dx \leq \left(\int_a^b f^\lambda\, dx\right)^{\frac{1}{\lambda}} \left(\int_a^b g^{-\mu}\, dx\right)^{\frac{1}{\mu}}.$$

This is known as Hölder's inequality.