REAL ANALYSIS
CARDINAL NUMBERS

We use $\overline{\overline{S}}$ for the cardinal number of a set S.

I $\overline{\overline{S}} \leq \overline{\overline{T}}$ (or $\overline{\overline{T}} \geq \overline{\overline{S}}$) is to mean "$\exists$ a 1-1 correspondence between S and a subset of T" (not necessarily a proper subset).

II $\overline{\overline{S}} = \overline{\overline{T}}$ is to mean "$\exists$ a 1-1 correspondence between S and T.

[$<$ is to mean $\leq$ but not $=$]

We have that:

(i) The definitions are reasonable when applied to finite sets.

(ii) **(a)** $\leq$ is transitive, i.e.

$$\overline{\overline{X}} \leq \overline{\overline{Y}} \quad \overline{\overline{Y}} \leq \overline{\overline{Z}} \Rightarrow \overline{\overline{X}} \leq \overline{\overline{Z}}$$

(b) $=$ is transitive

$$\overline{\overline{X}} = \overline{\overline{Y}} \quad \overline{\overline{Y}} = \overline{\overline{Z}} \Rightarrow \overline{\overline{X}} = \overline{\overline{Z}}$$

$=$ is symmetric

$$\overline{\overline{S}} = \overline{\overline{T}} \Leftrightarrow \overline{\overline{T}} = \overline{\overline{S}}$$

$=$ is reflexive
$\overline{\overline{S}} = \overline{\overline{S}}$

(iii) (Bernstein's Lemma) $\overline{\overline{S}} \leq \overline{\overline{T}}\ \overline{\overline{T}} \leq \overline{\overline{S}} \Rightarrow \overline{\overline{S}} = \overline{\overline{T}}$

(iv) For any two sets either $\overline{\overline{S}} \leq \overline{\overline{T}}$ or $\overline{\overline{T}} \leq \overline{\overline{S}}$.

A set S is said to be enumerable (denumerable, countable) $\Leftrightarrow \exists$ a 1-1 correspondence between S and the set of all natural numbers.
χ_0 is called the cardinal number of the set of all natural numbers.

1. If $\overline{\overline{S}} \leq \chi_0$ either S is finite or $\overline{\overline{S}} = \chi_0$
2. If $\overline{\overline{S}} = \chi_0$ S can be put in 1-1 correspondence with proper subset of itself.
3. Any infinite subset contains an enumerable subset.

4. If $\overline{\overline{S}} = \chi_0$ and T is infinite then $\overline{\overline{S \cup T}} = \overline{\overline{T}}$.
5. A set is infinite $\Leftrightarrow$ it can be put into 1-1 correspondence with a proper subset of itself.

Proof of A Suppose U and V are such that $\overline{\overline{U}} = \overline{\overline{V}} = \chi_0$.

Then $U = u_1 \ u_2 \ u_3 \dots$

$V = v_1 \ v_2 \ v_3 \dots$

$U \cup V = u_1 \ v_2 \ u_2 \ v_2 \dots = W = w_1 \ w_2 \ w_3 \ w_4 \dots$ therefore $\overline{\overline{U \cup V}} = \chi_0$.

We define a 1-1 correspondence between S and T thus T contains a subset $T' | \overline{\overline{T}}' = \chi_0$.

We map $S \cup T'$ onto T'(1-1) and map all the elements of T not in T' onto themselves therefore $\overline{\overline{S}} \cup \overline{\overline{T}} = \overline{\overline{T}}$.

I The set of all pairs (m, n) of all natural numbers is enumerable.

Set up the 1-1 correspondence $(m, n) \Leftrightarrow 2^m 3^n$, for by the theorem of uniqueness of prime factorisation $2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2} \Leftrightarrow m_1 = m_2 \ n_1 = n_2$.

Therefore we have mapped the set onto an infinite subset of the natural numbers which is enumerable.

II $S_1, S_2, S_3, \dots$ enumerable $\Rightarrow U_{r=1}^{\infty} S_r$ enumerable.

$S_1 = a_{11} \ a_{12} \ a_{13} \dots$

$S_2 = a_{21} \ a_{22} \ a_{23} \dots$

We assign to a_{mn} the number given by :

$f(mn)$ is a 1-1 correspondence between the set of pairs of natural numbers and the natural numbers. Assign to an element x of Us_r the least natural number of $f(mn)$ for which $a_{mn} = x$. Then Us_r is enumerable.

III The set of integers h is enumerable.

The set of natural numbers q is enumerable.

The set of all pairs (h, q) is enumerable by (2) $\exists f(h, q)$ mapping the pairs (h, q) in the natural numbers. Now to any rational r assign the least $f(h, q)$ for which $r = \frac{h}{q}$.

IV Suppose $X^1 \ X^2 \dots X^n$ are enumerable sets. Then the set of $(x^{(1)} x^{(2)} \dots x^{(n)}$ where each $x^{(r)}$ runs independently through the elements of X_r, is enumerable.

$\exists$ a 1-1 correspondence between the $x^{(r)}$ and the natural numbers U_u.

We use induction. Suppose true for $n = m$. $\exists$ a 1-1 correspondence between $(U_1 \dots U_m\ U_{m+1})$ and (V, U_{m+1}).

This is enumerable by III.

V The result of IV remains true if we have an enumerable system $X^{(1)}X^{(2)}\dots$ and consider all $(x^{(1)}x^{(2)}\dots x^{(n)})$ with n variables but finite.

We have $S : (x^{(1)}x^{(2)}\dots x^{(n)})$

We have $S_n : (x^{(1)}x^{(2)}\dots x^{(n)}$, n fixed. This is enumerable by IV.

$S = \cup_{r=1}^{\infty} S_n$ and is enumerable by II.

VI Consider the set of all polynomials.

$S = b_0x^n + b_1x^{n-1} + \dots + b_n$ where the b_i are integers. This set is enumerable by V, but to each $P_n(x)$ corresponds at most n algebraic numbers. Hence the set of all algebraic numbers is enumerable.

Example The set of discontinuities of a given monotone function is enumerable.

VII The set of all real numbers is not enumerable.

(i) Consider all real numbers $0 < \alpha \leq 1$. Each α has a unique decimal expansion

$$.x_1x_2x)3\dots$$

providing we insist that the number of non-zero x's is not finite.

Suppose the set of $alpha$ in $0 < \alpha \leq 1$ is enumerable. Enumerate them

$$\begin{aligned} \alpha_1 &+ .x_{11}x_{12}x_{13}x_{14}\dots \\ \alpha_2 &= .x_{21}x_{22}x_{23}x_{24}\dots \\ \alpha_3 &= .x_{31}x_{32}x_{33}x_{34}\dots \end{aligned}$$

Let $\beta = .y_1y_2y_3\dots\ \ 0 < \beta \leq 1$ where $y_r = 1$ if $x_{rr} \neq 1$ and $y_r = 2$ if $x_{rr} = 1$.

This β is not to be found in the sequence $\alpha_1\alpha_2\dots$. $\beta \neq \alpha_n$ since they differ in the nth place.

(ii) Let $0 \leq \alpha \leq 1$ and suppose that $\alpha_1 \alpha_2$ is an enumeration of this set.

Trisect the interval [01] by 3 closed intervals. At least one does not contain α_1. Choose J_1, the interval nearest the left not containing α_1. $\exists J_2$ nearest the left not containing α_2. These intervals tend to a limit point l $0 \leq L \leq 1$. For if $J_1 = [a_n\ b_n]$ $a_n \rightarrow l$ $b_n \rightarrow l$ as $n \rightarrow \infty$.

$a_1 \notin J_1,\ \alpha_2 \notin J_2 \ldots \quad J_1 \supset J_2 \supset J_3 \ldots$ therefore

$l \in J_n$ for $n = 1, 2, \ldots$ therefore

$l \neq \alpha_n \ldots$

Contradiction- for $0 \leq l \leq 1$.

VIII For any $a < b$ the points of (ab) have the same cardinal number as the points in (0 1) $\exists$ a 1-1 correspondence between the points of (0 1) and the points of $(a\ b)$ given by $0 < t < 1$ $t \leftrightarrow a + (b - a)t$.

IX The cardinal number of the set of real numbers is the same as the cardinal number of the set of points (0 1).

A (1-1) correspondence is $y \leftrightarrow \tanh(x) \quad -\infty < x < +\infty \quad -1 < y < 1$

We denote the cardinal number of the set of real numbers by C.

X The set of points in R_2 has cardinal C $(x\ y) \leftrightarrow \left(\frac{1+\tanh x}{2}, \frac{1+\tanh y}{2}\right)$ maps the points of R_2 onto the open square $0 < x < 1$ $0 < y < 1$.

Consider the point $(U\ V)$

$U = .x_1 x_2 \ldots,\ v = .y_1 y_2 \ldots$, unique if we exclude terminating decimals.

$$(UV) \leftrightarrow \alpha = .x_1 y_1 x_2 y_2 \ldots$$

If $(U_1 V_1)$ $(U_2 V_2)$ are different

$$(U_1 V_1) \leftrightarrow \alpha_1\ (U_2 V_2) \leftrightarrow \alpha_2\ \ alpha_1 \neq \alpha_2.$$

We have set up a 1-1 correspondence between the set T of points of the square, and a subset S of the points of (0 1). We can map the points of S onto a subset of T by $U \leftrightarrow (U, \frac{1}{2})$. Therefore $\overline{\overline{T}} \leq C \quad c \leq \overline{\overline{T}}$ therefore $\overline{\overline{T}} = c$

XI We can extend the result of X to R_n by induction.

Example The set of all sequences $x_1x_2x_3\dots$ of real numbers has cardinal C.