REAL ANALYSIS
PARTIAL SUMMATION (ABEL SUMMATION)

As part of the analogy existing between summation and integration, partial summation corresponds to integration by parts.

If $u \leq v$ and $s_m = \sum_{r=u}^{m} a_r$ then we have the identity

$$\sum_{m=u}^{v} a_m b_m = b_{v+1} s_V + \sum_{m=u}^{v} s_m (b_m - b_{m+1}) \tag{1}$$

Proof

$$\begin{aligned} \sum_{m=u}^{v} a_m b_m &= \sum_{m=u}^{v} (s_m - s_{m-1}) b_m \\ &= b_{v+1} s_v + \sum_{m=u}^{v} s_m (b_m - b_{m+1}) \end{aligned}$$

with the convention that empty sums are zero.

Abel's lemma With the above notation, suppose that $\{b_m\}$ is a positive monotonic decreasing sequence, and that $|s_m| \leq M$ for all m.

Then

$$\left| \sum_{m=u}^{v} a_m b_m \right| \leq M b_v$$

Proof

$$\begin{aligned} \left| \sum_{m=u}^{v} a_m b_m \right| &= \left| \sum_{m=u}^{v} s_m (b_m - b_{m+1}) + s_v b_{v+1} \right| \\ &\leq \sum_{m=u}^{v} |s_m| (b_m - b_{m+1}) + |s_v| b_{v+1} \\ &\leq M \left[\sum_{m=u}^{v} (b_m - b_{m+1}) + b_{v+1} \right] \\ &= M b_0 \end{aligned}$$

Theorem 6 Dirichlet's test Suppose that ϕ_n is a monotonic decreasing sequence converging to zero, and that $\sum a_n$ is a series with bounded partial sums. Then $\sum_{n=1}^{\infty} a_n \phi_n$ is convergent.

Proof $\left|\sum_{m=1}^{n} a_m\right| < K$ for all n

$$\left|\sum_{m=0}^{v} a_m\right| = \left|\sum_{1}^{v} a_m - \sum_{1}^{u-1} a_m\right| \leq \left|\sum_{1}^{v} a_m\right| + \left|\sum_{1}^{u-1} a_m\right| < 2K.$$

Given $\varepsilon > 0$, $\exists$ a natural number $N = N(\varepsilon)|phi_v < \frac{\varepsilon}{2K}$ for all $u \geq N$. By Abel's Lemma, therefore, $\left|\sum_{m=0}^{v} a_m\phi_m\right| \leq 2K\left(\frac{\varepsilon}{2K}\right) = \varepsilon$ whenever $v \geq u \geq N \Rightarrow \sum a_n\phi_n$ converges by general principle of convergence.

Theorem 7 Abel's Test Suppose that ϕ_n is a monotonic sequence converging to a finite limit. Let $\sum a_n$ be a convergent series. Then $\sum_{n=1}^{\infty} a_n\phi_n$ is convergent.

Proof

1. Suppose ϕ_n is monotonic decreasing and z
   $phi_n \to l$ as $m \to \infty$ therefore ψ_n is decreasing and $\psi_n = \phi_n - l \to 0$ as $n \to \infty$. Therefore by Direchlet's test $\sum a_n\psi_n$ converges. Write

$$\begin{aligned}\Psi &= \lim_{m\to\infty} \sum_{r=1}^{m} a_n(\phi_n - l) \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} a_n\phi_n - l\sum_{1}^{\infty} a_n\end{aligned}$$

   Therefore $\sum_{a=1}^{\infty} a_n\phi_n = \Psi - l\sum_1^{\infty} a_n$.

2. Suppose ϕ_n is monotonic increasing and $\phi_n \to L$ as $m \to \infty$. Write $\psi'_n = l - \phi_n$ ψ'_n is increasing and $\psi'_n \to 0$. Therefore as before $\sum a_n\phi_n$ converges.

Theorem 8 Root Test The series $\sum u_n$ converges or diverges according as $\overline{\lim}(u_n)^{\frac{1}{n}}$ is greater than or less than one.

Proof

1. Suppose $\overline{\lim}_{n\to\infty}(u_n)^{\frac{1}{n}} = \alpha < 1$. Choose $\beta|\alpha < \beta < 1$. Take $\varepsilon = \beta - \alpha > 0$. ¿From the property of the upper limit, $\exists m = m(\beta)|(u_n)^{\frac{1}{n}} < \beta$ for all $n \geq m$ so $u_n < \beta^n$ for all $n \geq m$. Therefore $\sum u_n$ converges by comparison with $\sum \beta^n$.
2. Suppose $\overline{\lim}_{n\to\infty}(u_n)^{\frac{1}{n}} = \alpha > 1$ then $(u_n)^{\frac{1}{n}} > 1$ for an infinity of n therefore $u_n > 1$ for an infinity of n therefore $u_n \not\to 0$ as $n \to \infty$ therefore $\sum u_n$ diverges.

Theorem 9 $\exists$ a number R such that the power series $\sum a_n z^n$ converges absolutely for $|z| < R$ and diverges for $|z| > R$, and $R^{-1} = \overline{\lim}_{n\to\infty}|a_n|^{\frac{1}{n}}$, with the appropriate conventions when RHS=0 or $+\infty$.

Proof **(i)** if $|a_n|^{\frac{1}{n}} \to 0$ as $n \to \infty$ $|a_n z^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}|z| \to 0$ as $n \to \infty$ for all z therefore by Root rest $\sum |a_n z^n|$ converges.

(ii) If $\overline{\lim}|a_n|^{\frac{1}{n}} = \infty$ the power series does not converge for $z \neq 0$ since $\overline{\lim}|a_n z^n|^{\frac{1}{n}} = \overline{\lim}|a_n|^{\frac{1}{n}} R = +\infty$.

(iii) If $\overline{\lim}|a_n|^{\frac{1}{n}}$ is finite and non-zero, we write it equal to $\frac{1}{R}$ $R > 0$ $\overline{\lim}|a_n z^n|^{\frac{1}{n}} = \frac{|z|}{R}$. Hence by root test, the series converges or diverges according as $|z| < R$ or $|z| > R$. R is called the radius of convergence.

$R^{-1} = \overline{\lim}|a_n|^{\frac{1}{n}}$ with, conventionally, $R = 0$ if RHS$= +\infty$ and $R = \infty$ is RHS=0.