Question

A particle is projected vertically upward in a constant gravitational field with an initial speed of v_0. Show that if there is a retarding force proportional to the square of the speed, the speed of the particle when it returns to its initial position is

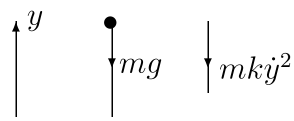
$$\frac{v_0 v_T}{\sqrt{v_0^2 + v_T^2}},$$

where v_T is the terminal speed.

Answer

Do this question in 2 parts.

Upward motion



Using Newton's 2nd law gives:

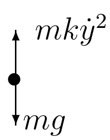
$$\begin{aligned} m\ddot{y} &= -mk\dot{y}^2 - mg \\ \frac{dv}{dt} &= -kv^2 - g \quad \text{Put } \frac{dv}{dt} = v\frac{dv}{dy} \\ \int \frac{v\,dv}{kv^2 + Y} &= -y + C \\ \text{whence } y &= \frac{1}{2k} \ln\left(\frac{kv_0^2 + g}{kv^2 + g}\right), \end{aligned}$$

where v_0 is the initial upwards velocity.

At the highest point $v = 0$.

Height attained $= \dfrac{1}{2k} \ln\left(\dfrac{k_0^2 + g}{g}\right)$

Downward motion



Using Newton's 2nd law gives:

$$\begin{aligned} \ddot{y} &= k\dot{y}^2 - g \\ y &= \frac{1}{2k}\ln\left(\frac{g-kv^2}{g}\right) + \frac{1}{2k}\ln\left(\frac{kv_0^2+g}{g}\right), \quad (*) \end{aligned}$$

where $v = 0$ at the highest point.
Solving for v gives $v = \sqrt{\frac{\frac{g}{k}v_0^2}{v_0^2+\frac{g}{k}}}$
Terminal velocity occurs when there is no net force on the mass while the particle is falling i.e. $mkv_t^2 = mg \Rightarrow v_t = \sqrt{\dfrac{g}{k}}$.

$$\text{Therefore} \quad v = \frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$