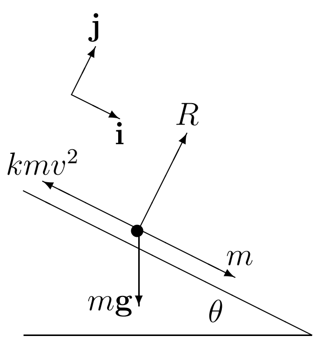
Question

A particle of mass m slides down an inclined plane under the influence of gravity. If the motion is resisted by a force $f = kmv^2$, show that the time required to move a distance d starting from rest is

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg\sin\theta}},$$

where θ is the inclination of the plane to the horizontal.

Answer



Use the **i** and **j** aligned with the slope and the reaction.
Using Newton's 2nd law gives:

$$\begin{aligned}
m\ddot{x}\mathbf{i} &= -kmv^2\mathbf{i} + R\mathbf{j} \\
&\quad +mg(\mathbf{i}\sin\theta - \mathbf{j}\cos\theta) \\
m\ddot{x} &= -km\dot{x}^2 + mg\sin\theta \\
\ddot{x} &= -k\dot{x}^2 + g\sin\theta \\
\text{put } \dot{x} = v \qquad v\frac{dv}{dx} &= -kv^2 + s\sin\theta \\
\int \frac{v\,dv}{g\sin\theta - kv^2} &= x + A \\
-\frac{1}{2k}\ln|g\sin\theta - kv^2| &= x + A
\end{aligned}$$

Substituting the initial conditions of $v = 0$ when $x = 0$ to find A:

$$\begin{aligned}
A &= -\frac{1}{2k}\ln|g\sin\theta| \\
\Rightarrow \qquad -\frac{1}{2k}\ln|g\sin\theta - kv^2| &= x - \frac{1}{2k}\ln|g\sin\theta|
\end{aligned}$$

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\begin{aligned}
\ln|g\sin\theta| &= -2kx + \ln|g\sin\theta| \\
g\sin\theta - kv^2 &= e^{-2kx+ln|g\sin\theta|} \\
-kv^2 &= g\sin\theta(e^{-2kx}-1) \\
v^2 &= \frac{g}{k}\sin\theta(1-e^{-2kx}) \\
\Rightarrow v &= \sqrt{\frac{g\sin\theta}{k}}\sqrt{(1-e^{-2kx})} \\
\text{substituting back for} \quad \dot{x} = v & \\
\Rightarrow \int \frac{dx}{\sqrt{1-e^{-2kx}}} &= t\sqrt{\frac{g\sin\theta}{k}} + const. \\
\text{substitute } u = e^{kx} & \quad \Rightarrow du = ku\,dx \\
\text{integral becomes } \int \frac{k^{-1}u^{-1}}{\sqrt{1-u^{-2}}} &= \frac{1}{k}\int \frac{du}{\sqrt{u^2-1}} = \frac{1}{k}\cosh -1 u \\
\text{Hence } \frac{1}{k}\cosh^{-1}(e^{kx}) &= t\sqrt{\frac{g\sin\theta}{k}} + const \\
\text{Hence on rearranging this result} & \\
t &= \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg\sin\theta}}
\end{aligned}
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