

### Question

Find the solution of the initial value problem

$$x^2 t \frac{dx}{dt} = x^3 + t^3$$

subject to  $x(1) = 0$

### Answer

$$x^2 t \frac{dx}{dt} = x^3 + t^3 \Rightarrow \frac{dx}{dt} = \frac{x}{t} + \left(\frac{t}{x}\right)^2 = f\left(\frac{x}{t}\right)$$

This is of the form  $y = \frac{x}{t} \Rightarrow \frac{dx}{dt} = t \frac{dy}{dt} + y = y + \frac{1}{y^2} \Rightarrow t \frac{dy}{dt} = \frac{1}{y^2}$

$$\text{Thus } \int y^2 dy = \int \frac{dt}{t} \Rightarrow \frac{1}{3} y^3 = \ln |t| + \text{constant}$$

Now apply the initial condition  $x(1) = 0$  for  $t > 1$  so  $|t| = t$

$$y(1) = \frac{x(1)}{t=1} = x(1) \text{ and } \ln 1 = 0 \Rightarrow \text{constant} = 0$$

Therefore

$$x^3 = 3 \ln t \Rightarrow x = \sqrt[3]{3 \ln t}$$