Question

Find the general solution of the differential equation

$$t\frac{dx}{dt} = x + \frac{1}{2}t\sec^2\frac{x}{2t}$$

Answer

$t\dfrac{dx}{dt} = x + \dfrac{1}{2}t\sec^2\dfrac{x}{2t}$

Rewrite as $\dfrac{dx}{dt} = \dfrac{x}{t} + \dfrac{1}{2}\sec^2\dfrac{x}{2t}$

This is of the form $\frac{dx}{dt} = f\left(\frac{x}{t}\right)$ So let $y = \frac{x}{t}$

$\Rightarrow \frac{dx}{dt} = t\frac{dy}{dt} + y = y + \frac{1}{2}\sec^2\frac{1}{2}y$

So we can rewrite as

$$t\frac{dy}{dt} = \frac{1}{2}\sec^2\frac{1}{2}y$$

Cross Multiply

$$\frac{dt}{t} = \frac{2}{\sec^2\frac{1}{2}y}dy = 2\cos^2\frac{1}{2}y\,dy$$

Now $2\cos^2\frac{y}{2} = 1 + \cos y$ so the differential equation becomes

$$\int\frac{dt}{t} = \int(1+\cos y)\ dy$$

Integrating

$$\ln|t| = y + \sin y + \text{constant}$$

$$t = Ae^{y+\sin y}$$

with A as a constant.