Question

The wave function for the ground state of the hydrogen atom is given by

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{\frac{-r}{a_0}}.$$

The probability of finding the electron at a distance r from the nucleus is given by

$$p(r) = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} (\psi(r,\theta,\phi))^2 r^2 \sin(\theta)\, d\theta\, d\phi.$$

Calculate $p(r)$ and show that it has a maximum when $r = a_0$.

Answer

$$\begin{aligned}
p(r) &= \frac{1}{\pi a_0^3} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta e^{\frac{-2r}{a_0}}\, d\theta\, d\phi \\
&= \frac{r^2 e^{\frac{-2r}{a_0}}}{\pi a_0^3} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \\
&= \frac{r^2 e^{\frac{-2r}{a_0}}}{\pi a_0^3} [\phi]_0^{2\pi} [-\cos\theta]_0^{\pi} \\
&= \frac{r^2 e^{\frac{-2r}{a_0}}}{\pi a_0^3} (2\pi)(1+1) \\
&= \frac{4r^2 e^{\frac{-2r}{a_0}}}{a_0^3}
\end{aligned}$$

To find stationary points, calculate $\frac{\partial p}{\partial r}$. Using chain rule:

$$\frac{\partial p}{\partial r} = \frac{8re^{\frac{-2r}{a_0}}}{a_0^3} - \frac{8r^2 e^{\frac{-2r}{a_0}}}{a_0^4} = \frac{8r(a_0 - r)e^{-\frac{2r}{a_0}}}{a_0^4}$$

So $\frac{\partial p}{\partial r} = 0$ when $r = 0$ or $r = a_0$ $\left(e^{\frac{-2r}{a_0}} \neq 0\right)$

$\underline{r = 0}$ As r increases through $r = 0$, $\frac{\partial p}{\partial r}$ changes from negative to positive. So $p(r)$ has a minimum at $r = 0$.

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$\underline{r = a_0}$ As r increases through $r = a_0$, $\frac{\partial p}{\partial r}$ changes from positive to negative. So $p(r)$ has a maximum at $r = a_0$.

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