**Question**

The following equations are written in terms of spherical polar co-ordinates $(r, \theta, \phi)$. What surfaces or curves do they represent?

**(a)** $r\cos(\theta) = 1$;

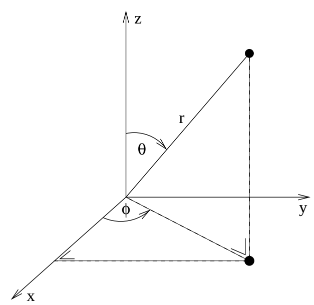
**(b)** $\sin(\theta) = \frac{\pi}{4}$;

**(c)** $\theta = \frac{\pi}{2}$, $r\cos(\phi) = 0$;

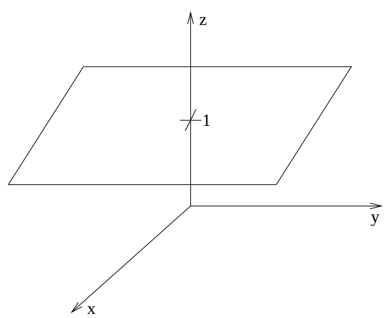
**(d)** $\theta = \frac{\pi}{4}$, $r\cos(\theta) = 1$.

**Answer**

Spherical polar co-ordinates $(r, \theta, \phi)$.
$r \geq 0,\ 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$
with $\mathrm{x} = \mathrm{r}\sin\theta\cos\phi, \mathrm{y} = \mathrm{r}\sin\theta\sin\phi$
and $\mathrm{z} = \mathrm{r}\cos\theta$.

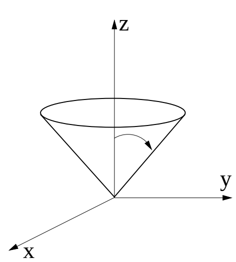



**(a)** $r\cos(\theta) = 1 \Rightarrow z = 1$ Since x and y are arbitrary, we have the plane parallel to the xy-plane at height z = 1.

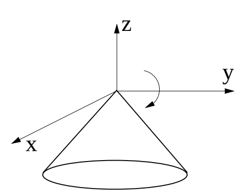



**(b)** $\sin(\theta) = \frac{\pi}{4}$. Since $0 \leq \theta \leq \pi$ there are two solutions, $\theta \approx 51.8°$ or $\theta = 180 - 51.8 = 128.2°$. Each of these gives a cone:

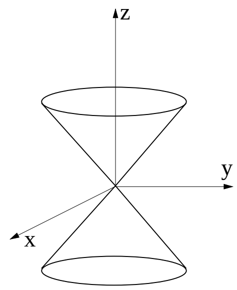
<u>$\theta = 51.8^o$</u>




<u>$\theta = 128.2^o$</u>



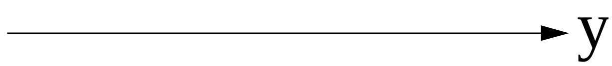
The required curve / surface is the union of all possible solutions and so we obtain the double cone:




**(c)** $\theta = \frac{\pi}{2} \Rightarrow z = r\cos\frac{\pi}{2} = 0$, so the surface / curve lies in the xy-plane. $r\cos(\phi) = 0 \Rightarrow$ either $r = 0$ or $\cos\phi = 0$

$\underline{r = 0}$ $(x, y, z) = (0, 0, 0)$ so we have the single point, the origin

$\underline{\cos\phi = 0, \quad (r \neq 0)}$ $x = r\sin\theta\cos\phi = 0$ and $\cos\phi = 0 \Rightarrow \sin\phi = +1$ or $-1$ so the required curve is just the y-axis.




**(d)** If $\theta = \frac{\pi}{4}$, $r\cos(\theta) = 1 \Rightarrow r\cos\frac{\pi}{4} = 1 \Rightarrow r = \sqrt{2}$.

Hence $z = r\cos\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$ This gives the circle lying in the plane $z = 1$, whose centre lies on the the z-axis. The circle has radius $r\sin\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$.