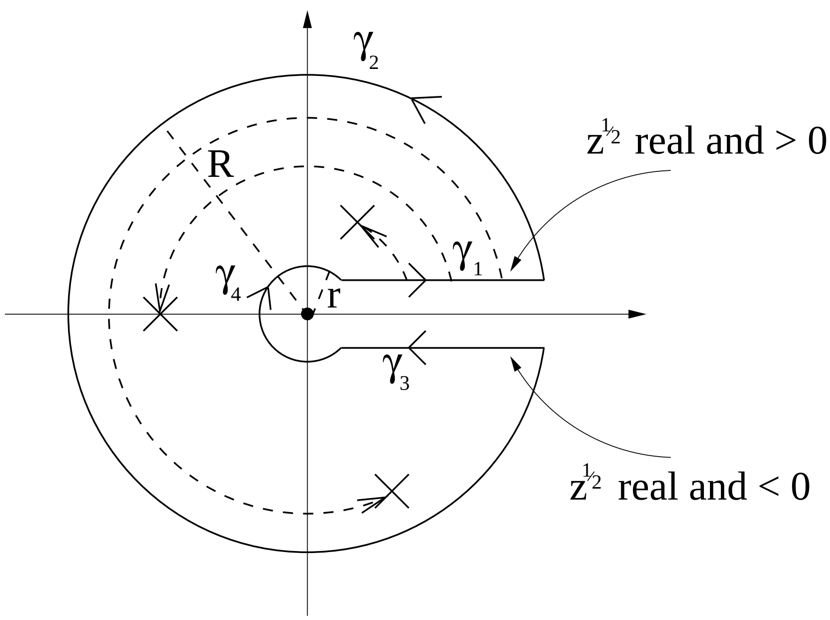
QUESTION Use the keyhole contour to evaluate

$$\int_0^\infty \frac{x^{\frac{1}{2}} dx}{1+x^3}$$

ANSWER

$$I = \int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^3}\, dx, \quad J = \int_C \frac{z^{\frac{1}{2}}}{1+z^3}\, dz$$



Simple poles at $z_0 = 1, e^{\pm i\frac{\pi}{3}}$

$\text{Res}\,(z_0) = \dfrac{z_0^{\frac{1}{2}}}{3z_0^2} = \dfrac{z_0^{-\frac{1}{2}}}{3z_0^3} = -\dfrac{1}{3} z_0^{-\frac{1}{2}}$

$\displaystyle\int_{\gamma_1} = I, \; \left|\int_{\gamma_2}\right| \le \frac{R^{\frac{1}{2}}}{R^3-1}\pi R \sim R^{-\frac{3}{2}} \to 0 \text{ as} R \to \infty$

$\displaystyle\int_{\gamma_3} = \int_\infty^0 \frac{-x^{\frac{1}{2}}}{1+x^3}\, dx = I$

$\displaystyle\left|\int_{\gamma_4}\right| \le \frac{r^{\frac{1}{2}}}{1-r^3}\pi r \sim r^{\frac{3}{2}} \to 0 \text{ as } r \to 0$

$\displaystyle\Rightarrow J = 2I = 2\pi i\left(-\frac{1}{3}\right)\left(\left(e^{\frac{i\pi}{3}}\right)^{\frac{1}{2}} + \left(e^{i\pi}\right)^{\frac{1}{2}} + \left(e^{\frac{5i\pi}{3}}\right)^{\frac{1}{2}}\right)$

$\displaystyle= -\frac{2\pi i}{3}\left(e^{\frac{i\pi}{6}} + i - e^{-\frac{i\pi}{6}}\right) = \frac{2\pi}{3} + \frac{4\pi}{3}\sin\frac{\pi}{6} = \frac{2\pi}{3} + \frac{4\pi}{3}\frac{1}{2} = \frac{\pi}{3}$

(Note that we measure all arguments from the positive real axis.)