Question

In this question you may assume a standard Black-Scholes world in which there are no dividend yields, and hence $q = 0$.

Suppose that an option, W, is written which is worthless on expiry, but which pays out a continuous cash-flow of $K(S,t)dt$ during the time interval $(t, t+dt)$, prior to expiry. Using the standard Black-Scholes analysis, including the cash-flow Kdt, show that W satisfies the problem

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + rS\frac{\partial W}{\partial S} - rW = -K(S,t),$$

$$t < T, \qquad W(S,T) = 0.$$

Use financial arguments to show that if $K > 0$ for $t < T$ the $W > 0$ for $t < T$ and that if $K < 0$ for $t < T$ then $W < 0$ for $t < T$.

What is meant by "implied volatility" of an option V? What is the vega, φ,of an option, and how does it relate to the "implied volatility"?

Assuming that the payoff for the option is independent of volatility, show that the vega satisfies the problem

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \varphi}{\partial \S^2} + rS\frac{\partial \varphi}{\partial S} - r\varphi = -\sigma S^2 \frac{\partial^2 V}{\partial S^2}$$

$$\varphi(S,T) = 0$$

Hence deduce that "implied volatility" is well defined if the gamma of the option does not change sign.

Answer

Since W expires worthless, $W(S,T) = 0$.

Set up standard portfolio $\Pi = W + \Delta S$ where Δ is previsible, i.e. constant over interval dt, and assume usual geometric Brownian motion.

$$\frac{dS}{S} = \mu dt + \sigma dX \quad \text{where } \ \mathrm{dX}^2 = \mathrm{dt}$$

Then Itô implies

$$\begin{aligned} dW &= (W - t + \frac{1}{2}\sigma^2 S^2 W_{SS})dt + W_S dS \\ \text{and} \quad d\Pi &= dW - \Delta dS + K dt, \end{aligned}$$

since holding the option gives Kdt in cash.

Thus $d\Pi = (W_t + \frac{1}{2}\sigma^2 S^2 W_{SS})dt + (W_S - \Delta)dS + Kdt$.

Only risk comes in through dS term, so taking $\Delta = W_S$ eliminates it.As this makes Π riskless, its return must equal risk free rate so

$$d\Pi = \left(W_t + \frac{1}{2}\sigma^2 S^2 W_{SS} + K\right) dt = r\Pi dt = r(W - SW_S)dt$$

$$\Rightarrow W_t + \frac{1}{2}\sigma^2 S^2 W_{SS} + rSW_S - rW = -K$$

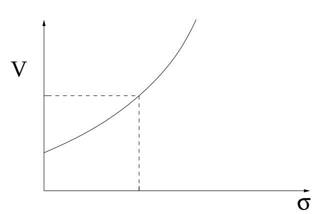
If $K > 0$ then holder of the option receives a guaranteed positive cash flow for $t < T \Rightarrow \ W > 0$ for $t < T$ (i.e. you'd pay a positive amount to get this positive cash flow)

If $K < 0$ then holder of the option has to pay out $-Kdt > 0$ every interval dt prior to expiry $\Rightarrow \ W < 0$ for $t < T$ (i.e. you would pay a positive amount to escape having to pay out K)

Implied vol is the volatility you would have to put into the Black-Scholes model to obtain an observed option price; given the option price and all other parameters ($S, \ r, \ K, \ t, \ T$ in this case)

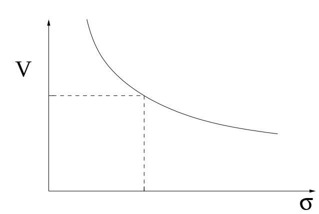
Vega $= \varphi = \frac{\partial V}{\partial \sigma}$, i.e. the sensitivity of the option price to small changes in vol, σ. ($dV = \frac{\partial V}{\partial \sigma} d\sigma$, all other parameters being fixed)

Implied vol is only uniquely defined if $\frac{\partial V}{\partial \sigma}$ does not change sign; i.e. can find σ as a function of V only if graph of V against σ is either



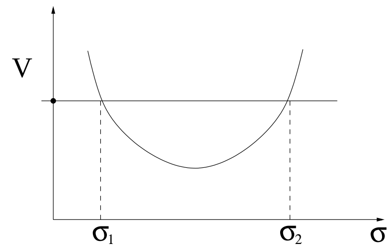
($\varphi = \frac{\partial V}{\partial \sigma} > 0$)

or



($\varphi = \frac{\partial V}{\partial \sigma} < 0$)

but if $\varphi = \frac{\partial V}{\partial \sigma}$ changes sign we must have something like



and $\sigma_1,\ \sigma_2$ both give same value of V

Take $\dfrac{\partial}{\partial \sigma}$ of Black-Scholes;

$$\begin{aligned} \frac{\partial}{\partial \sigma}\left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV\right) &= 0 \\ \Rightarrow \varphi_t + \frac{1}{2}\sigma^2 S^2 \varphi_{SS} + rS\varphi_S - r\varphi &= -\sigma S^2 V_{SS} \end{aligned}$$

and

$$\frac{\partial}{\partial \sigma}(V(S,T) = \text{something indep of } \sigma) \Rightarrow \varphi(\mathrm{S},\mathrm{T}) = 0.$$

From argument about W, we see that if $-\sigma S^2 V_{SS} > 0$, we must have $\varphi = \frac{\partial V}{\partial \sigma} > 0$ for $t < T$, so implied vol is well defined.
Likewise, if $-\sigma S^2 V_{SS} < 0$, we have $\varphi = \frac{\partial V}{\partial \sigma} < 0$ for $t < T$, so implied vol is well defined.
Now note that $S^2 > 0$, $\sigma > 0$ by definitions and the gamma, Γ, of an option is defined by $\Gamma = V_{SS}$.
Hence the result.