**Question**

Show that if $V(S,t)$ is a solution of the Black-Scholes equation with zero dividend yield, $q = 0$, then so too is $S^{\alpha}V(a/S,t)$, where $a$ and $\alpha$ are constants, provided $\alpha = 1 - 2r/\sigma^2$.

A down-and-out barrier call option is an option which has the same payoff as a call, but which becomes worthless if the spot price $S$ ever drops below a fixed barrier level $B$ (even if it subsequently rises back above $B$, the option remains worthless). Write down the Black-Scholes problem satisfied by the barrier call.

Explain why you would expect the value of this option to be less than that of an otherwise identical vanilla call option (without the barrier).

Assuming that the barrier lies below the strike, $B < K$, use the Black-Scholes formula for a call value and the above observation to obtain an exact formula for the barrier call option. [Hint, choose $a$ above so that $a/S = B$ when $S = B$.] Confirm that the barrier option is less valuable than an otherwise identical call option.

**Answer**

Let $U(S,t) = S^\alpha V(\frac{a}{S}, t)$ and note that since

$$\begin{aligned} V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(S,t) + rSV_S(S,t) - rV(S,t) &= 0 \\ \Rightarrow V_t\left(\frac{a}{S},t\right) + \frac{1}{2}\sigma^2 S^2 \frac{a^2}{S^2} V_{SS}\left(\frac{a}{S},t\right) + r\frac{a}{S}V_S\left(\frac{a}{S},t\right) & \\ -rV\left(\frac{a}{S},t\right) &= 0 \qquad (*) \end{aligned}$$

(N.B. $V_S$ means derivative wrt 1st argument, $V_t$ means derivative wrt 2nd argument)

Then

$$\left.\begin{aligned} U_t &= S^\alpha V_t \\ U_S &= \alpha S^{\alpha-1} V - aS^{\alpha-2} V_S \\ U_{SS} &= \alpha(\alpha-1)S^{\alpha-2}V - 2a(\alpha-1)S^{\alpha-3}V_S + a^2 S^{\alpha-4} V_{SS} \end{aligned}\right\}$$

All $V$'s evaluated at $(\frac{a}{S}, t)$

$$\begin{aligned} & U_t + \frac{1}{2}\sigma^2 S^2 U_{SS} + rSU_S - rU \\ = \;& S^\alpha \left\{ \begin{array}{l} V_t + \frac{1}{2}\sigma^2(\alpha(\alpha-1)V - 2(\alpha-1)\frac{a}{S}V_S + \frac{a^2}{S^2}V_{SS}) \\ +r(\alpha V - \frac{a}{S}V_S) - rV \end{array} \right\} \\ = \;& S^\alpha \left\{ \begin{array}{l} V_t + \frac{1}{2}\sigma^2 \frac{a^2}{S^2} V_{SS} + \frac{a}{S}(-r - (\alpha-1)\sigma^2)V_S \\ +(r(\alpha-1) + \frac{1}{2}\alpha(\alpha-1)\sigma^2)V \end{array} \right\} \\ = \;& 0 \quad \text{if (using } (*)) \end{aligned}$$

$$-r - (\alpha-1)\sigma^2 = r \quad \text{and} \quad (\alpha-1)(\mathrm{r} + \frac{\alpha}{2}\sigma^2) = -\mathrm{r}$$

$$\begin{aligned} \Rightarrow \quad -(\alpha-1) &= \tfrac{2r}{\sigma^2} \\ \Rightarrow \quad \alpha &= 1 - \tfrac{2r}{\sigma^2} \end{aligned}$$

So

$$\begin{aligned} (\alpha-1)\left(r + \frac{\alpha}{2}\sigma^2\right)) &= -\frac{2r}{\sigma^2}\left(r + \frac{\sigma^2}{2}\left(1 - \frac{2r}{\sigma^2}\right)\right) \\ &= -\frac{2r}{\sigma^2}\left(\frac{\sigma^2}{2} + r - r\right) \\ &= -r \quad \text{as desired} \end{aligned}$$

Barrier satisfies

| | | |
|---|---|---|
| $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV$ | $t < T, \quad S > B$ | (1) |
| $V(S,T) = \max(\mathrm{S} - \mathrm{K}, 0)$ | $S > B$ | (2) |
| $V(B,t) = 0$ | $t \leq T.$ | (3) |

Expect this to be less valuable because it has some payoff as call but may become worthless prior to expiry whereas call can't.
Say $C(S,t)$ is call value so it satisfies

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0 \ \ \S > 0$$

$$C(S,T) = \mathrm{max(S - K, 0)}$$

And consider

$$U = \frac{S^\alpha}{B^\alpha} C\left(\frac{B^2}{S}, t\right)$$

which satisfies

$$U_t + \frac{1}{2}\sigma^2 S^2 U_{SS} + rSU_S - rU = 0$$

$$U(S,T) = \frac{S^\alpha}{B^\alpha}\mathrm{max}\left(\frac{\mathrm{B}^2}{\mathrm{S}} - \mathrm{K}, 0\right)$$

and most importantly $U(S,T) = 0$ if $S > B$ since $B < K$.
Thus

$$V(S,t) = C(S,t) - \left(\frac{S}{B}\right)^\alpha C\left(\frac{B^2}{S}, t\right)$$

satisfies (1) and (2) above. Then at S=B

$$V(B,t) = C(B,t) - \left(\frac{B}{B}\right)^\alpha C\left(\frac{B^2}{B}, t\right) = 0$$

so (3) is satisfied, thus barrier value is

$$V(S,t) = C(S,t) - \left(\frac{S}{B}\right)^\alpha C\left(\frac{B^2}{S}, t\right)$$

Since call value $C > 0$ for $t < T$,

$$\left(\frac{S}{B}\right)^\alpha C\left(\frac{B^2}{S}, t\right) > 0, \quad \text{for t} < \text{T}$$

Hence

$$\begin{aligned} V(S,t) &< C(S,t) \\ \text{Barrier value} &< \text{Call value} \end{aligned}$$