Question

In this question you must assume zero dividend yields, $q = 0$.
Show that $V = \alpha S$, where α does not depend on S or t, is a solution of the Black-Scholes equation.
Let $C(S, t; T_1, K)$ denote the Black-Scholes value of a European vanilla call option with strike K and expiry T_1 at time t and spot price S. Show that the value of an "at the money" call is proportional to the spot (or strike).
A (European) forward-start call is a call option whose strike is not known at the start of the contract but, rather, is agreed to be the spot price S_0 a given time T_0 prior to expiry T_1. The option can not be exercised prior to expiry T_1.
Show that the value v of a European forward-start call option is given by

$$V(S,t) = \begin{cases} \alpha S & \text{if } t \le T_0, \\ C(S,t;T_1,S_0) & \text{if } T_0 \le t \le T_1 \end{cases}$$

where S_0 is the spot price at time T_0 and α is a function, which you should determine, which depends only on $T_1 - T_0$, σ and r.
Briefly justify the Δ-hedging strategy implied by this price for $t \le T_0$.

Answer

BS equation is $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$

If $V = \alpha S$, $V_t = 0$, $V_{SS} = 0$, $V_S = 1$, so we get

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - RV = rS - rS = 0$$

$\Rightarrow \alpha S$ is a solution.

Value of call is

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

At the money means, $K = S$ hence $\log(K/S) = 0$ and

$$d_1 = d_1(T,t), \quad d_2 = d_2(T,t)$$

$$\left.\begin{aligned} C = &\ K(N(d_1(T,t)) - e^{-r(T-t)}N(d_2(T,t)) \\ \propto &\ K \end{aligned}\right\} \text{or replace } K \text{ by } S$$

Suppose S_0 is the spot price at T_0. For $t > T_0$ we know the strike, it is $K + S_0$ and hence

$$V(S,t) = C(S,t;T_1,S_0)$$

At time $t = T_0$ we have

$$V(S_0,T_0) = S_0\alpha(T_1,T_0) = \alpha S_0$$

Now solve the BS equation back from $t = T_0$, i.e. solve

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0,$$

$$V(S,T_0) = \alpha S$$

$\Rightarrow V(S,t) = \alpha S$ for $t < T_0$

α is given by substituting $K = S_0$, $t = T_0$ into usual formula for a call, i.e.

$$\alpha = S_0\left[N\left(\frac{(r+\frac{1}{2}\sigma^2)(T_1-T_0)}{\sigma\sqrt{T_1-T_0}}\right)\right.$$

$$\left.-e^{-r(T_1-T_0)}N\left(\frac{(r-\frac{1}{2}\sigma^2)(T_1-T_0)}{\sigma\sqrt{T_1-T_0}}\right)\right]$$

The Δ hedging strategy is to hold α assets up to T_0, where α is the number of assets you need to hedge the at-the-money call into which the option turns at T_0. i.e. You know that at T_0 you will need α assets, so you hold them from time $t = 0$.