Question

Show that the Black-Scholes equation remains invariant under the scaling $S' = \alpha S$ where $\alpha > 0$ is a constant.

A put option with strike K is written on an asset which pays out on a single, discrete yield q at time $t_d < T$, where T is the expiry date of the put. Explain why the spot price jumps from S to $(1-q)S$ as the dividend date is crossed, but the option price remains continuous. Denote the option price by $P(S,t;K,T)$.

Let $P_{BS}(S,t;K,T)$ denote the usual Black-Scholes value for a put option on an asset which pays no dividends and has strike K, expiry T. Show that

$$P(S,t;K,T) = \begin{cases} P_{BS}(S,t;K,T) & \text{if } t_d < t < T, \\ (1-q)P_{BS}(S,t;K/(1-q),T) & \text{if } 0 \geq t < t_d. \end{cases}$$

Answer
Irrelevant whether they do this assuming $q = 0$ or $q \neq 0$.
For $q = 0$, BS is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - q)SV_S - rV = 0$$

Now if $S' = \alpha S$ we have

$$\frac{\partial}{\partial S} = \frac{\partial S'}{\partial S}\frac{\partial}{\partial S'} = \alpha\frac{\partial}{\partial S'},$$

$$\text{so}\ \ \mathrm{S}\frac{\partial}{\partial \mathrm{S}} = \frac{1}{\alpha}\mathrm{S}'\alpha\frac{\partial}{\partial \mathrm{S}'} = \mathrm{S}'\frac{\partial}{\partial \mathrm{S}'}$$

Hence

$$\begin{aligned} S\frac{\partial}{\partial S}\left(S\frac{\partial}{\partial D}\right) &= S'\frac{\partial}{\partial S'}\left(S'\frac{\partial}{\partial S'}\right) \\ \Rightarrow S^2\frac{\partial^2}{\partial S^2} + S\frac{\partial}{\partial S} &= s'^2\frac{\partial^2}{\partial S'^2} + S'\frac{\partial}{\partial S'} \\ \Rightarrow S^2\frac{\partial^2}{\partial S^2} &= S'^2\frac{\partial^2}{\partial S'^2} \end{aligned}$$

Thus

$$V_t + \frac{1}{2}\sigma^2 S'^2 V_{S'S'} = (r - q)S'V_{S'} - RV = 0$$

i.e. equation is invariant.
At time t_d asset pays out a dividend of qS (that is what a dividend yield q means), with certainty. If spot price immediately after t_d is not $(1-q)S$ we could arbitrage situation; eg if spot is $\hat{S} > (1-q)S$ after t_d - cost is $(1-q)S$, selling yields $\hat{S} > (1-q)S$.
If spot is $\overline{S} < (1-q)S$ after t_d, buy asset before t_d, collect dividend and then sell for $\overline{S}$ $\Rightarrow$ risk free profit.
Option doesn't pay any cash dividends, so must have $V(t_d^-) = V(t_d^+)$.
Write this as

$$\begin{aligned} S &= S^- \text{ at } \mathrm{t_d^-} \\ S &= (1-q)S^- \text{ at } \mathrm{t_d^+} \\ V(S^-, t_d^-) &= V(S^+, t_d^+) \end{aligned}$$

$\Rightarrow$ jump condition

$$\begin{aligned} v(S^-, t_d^-) &= V(S^-(1-q), t_d^+) \ \text{ or just} \\ v(S, t_d^-) &= V(S(1-q), t_d^+). \end{aligned}$$

for $t > t_d$ we have (since $q = 0$ if there is only a DISCRETE dividend)

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - rSV_S - rV = 0,$$

$$V(S,T) = \max(\mathrm{K} - \mathrm{S}, 0)$$

By definition, the solution of this problem is $P_{BS}(S,t;K,T)$ so

$$V = P_{BS}(S,t;K,T) \;\; \text{for } \mathrm{t} > \mathrm{t_d}$$

Now write jump condition as $V(S,t_d^-) = V(S(1-q),t_d^+)$ so, at t_d^-

$$V(S,t_d^-) = P_{BS}((1-q)S,t_d^-;K,T)$$

Now consider payoff for $P_{BS}((1-q)S,t;K,T)$,
i.e. $P_{BS}((1-q)S,T;K,T)$.
It is

$$\begin{aligned} P_{BS}((1-q)S,T;K,T) &= \max(\mathrm{K} - (1-\mathrm{q})\mathrm{S}, 0) \\ &= (1-q)\max(\frac{\mathrm{K}}{1-\mathrm{q}} - \mathrm{S}, 0) \end{aligned}$$

Hence, since BS is invariant under $S \to (1-q)S$, is linear problem for $V(S,t),\ t < t_d$ is equivalent to

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0,$$

$$V(S,T) = (1-q)\max(\frac{\mathrm{K}}{1-\mathrm{q}} - \mathrm{S}, 0)$$

i.e.

$$V = (1-q)P_{BS}(S,t;\frac{K}{1-q},T)$$

Hence the result.