Question

Consider the simple binomial model for asset price changes:
The asset price at time $t = 0$ os $S + 0$ and at time $t = 1$ it can either be S_2 with the probability $0 < p < 1$ or $S_1 < S_2$ with probability $1 - p$. Show that unless

$$S_1 < S_0 e^r < S_2$$

where r is the risk free rate, this model is arbitragabile. (You may assume that short sales are allowed.)
A derivative security, V, is written on this asset. At time $t = 0$ its value is V_0. At time $t = 1$ its value can be either V_1, if the underlying's price is S_1, or V_2 if the underlying's price is S_2.

(a) Let V_0^p denote the present value (at time $t = 0$) of the expected value of V at time $t = 1$. Give a formula for V_0^p in terms of p, r, V_1 and V_2.

(b) Construct a risk free portfolio containing bot V and S and use an arbitrage argument to show that this leads to a "fair price" for V_0, say V_0^Δ, in terms of S_0, S_1, S_2, V_1 and V_2 but *not* p.

(c) Construct a replicating strategy, in terms of S and cash invested at the risk-free rate r, which leads to an arbitrage free price for V_0, say V_0^R, in terms of S_0, S_1, S_2, V_1 and V_2 but *not* p.

(d) Deduce from wither (b) or (c) that there is a number q, which may be regarded as a "risk neutral" probability, associated with the underlying's price, such that the "fair value" of V_0 is the present value (at $t = 0$) of the expected value of V at time $t = 1$.

(e) Assuming that either $V_2 < V_1$ or $V_2 > V_1$, and that $p > q$, show that a trader using either of the prices $V_0^\Delta = V_)^T$ from (b) or (c) would necessarily be able to arbitrage a trader using the price V_0^p from (a).

Answer

If $S_1,\ S_2 > S_0 e^r$ then whatever happens, risky asset grows faster than risk free rate $\Rightarrow$ borrow S_0 now buy risky asset, payback loan next time step $\Rightarrow$ arbitrage

If $S_1,\ S_2 < S_0 e^{rst}$, short sell asset now, invest in bank, use $S_0 e^r > S_1,\ S_2$ to close out short sale next time step $\Rightarrow$ arbitrage.

(a) $V_0^p = e^{-r}(pV_2 + (1-p)V_1)$

(b) $\Pi = V - \Delta S$ so $\Pi_0 = V_0 - \Delta S_0$.

At time 1, $\Pi = \begin{cases} \Pi_1 & if S_0 \to S_1 \\ \Pi_2 & if S_0 \to S_2 \end{cases}$

where $\Pi_1 = V_1 - \Delta S_1$, $\Pi_2 = V_2 - \Delta S_2$. Eliminate risk free by setting $\Pi_1 = \Pi_2$.

$$\begin{aligned} \Rightarrow V_1 - \Delta S_1 &= V_2 - \Delta S_2 \\ \Delta &= \frac{V_2 - V_1}{S_2 - S_1} \end{aligned}$$

This gives $\Pi_0 = e^{-r}\Pi_1 = e^{-r}\pi_2$ since Π is riskless and hence

$$\begin{aligned} V_0 &= e^{-r}\pi_1 \\ \Rightarrow V_0 &= e^{-r}(V_1 - \Delta S_1) + \Delta S_0. \end{aligned}$$

Price is "fair" in the sense that if you think true price is $\overline{V}_0 > V_0$, then soon can sell you option for $\overline{V}_0$, construct risk free portfolio and close out at $t = 1$, making $\overline{V}_0 - V_0$ risk free profit. Conversely, if someone is willing to sell for $\hat{V}_0 < V_0$, you should buy, short sell risk free portfolio and close out at $t = 1$ making $V_0 - \hat{V}_0 > 0$ risk free profit.

(c) Let $\Pi = \phi S_0 + \psi$ where ψ is invested at risk free rate.

$$\begin{aligned} \Pi_0 &= \phi S_0 + \psi, \\ \Pi_1 &= \phi S_1 + \psi e^r, \\ \Pi_2 &= \phi S_2 + \psi e^r. \end{aligned}$$

setting $\Pi_1 = V_1$, $\Pi_2 = V_2$, i.e.

$$\phi S_1 + \psi e^r = V_1, \quad \phi S_2 + \psi e^r V_2$$

$$\Rightarrow \phi = \frac{V_2 - V_1}{S_2 - S_1},\ \psi = e^{-r}(V_1 - \phi S_1) = e^{-r}(V - 2 - \phi S_2).$$

Then, since $\Pi = V$ at time 1, with certainty, $\Pi_0 = V_0$, hence

$$V_0 = e^{-r}(V_1 - \phi S_1) + \phi S_0$$

(note ϕ in (c) $= \Delta$ in (b))

(d) From either (b) or (c)

$$\begin{aligned}
V_0 &= e^{-r}(V_1 - \frac{V_2 - V_1}{S_2 - S_1}s_1) + \frac{V_2 - V_1}{S_2 - S_1}S_0 \\
&= \frac{1}{S_2 - S - 1}[e^{-r}(V - 1(S_2 - S_1) - (V_2 - V_1)S_1) \\
&\quad +(V_2 - V_1)S_0] \\
&= \frac{e^{-r}}{S_2 - S_1}[V_1S_2 - V_1S_1 - V_2S_1 + V_1S_1 + V_2S_0e^r - V_1S_0e^r] \\
&= \frac{e^{-r}}{S_2 - S_1}[(S_0e^r - S_1)V_2 + (S_2 - S_0e^r)V_1] \\
&= e^{-r}[qV_2 + (1-q)V_1]
\end{aligned}$$

$$q = \frac{S_0e^r - S_1}{S_2 - S_1}, \quad 1 - q = \frac{S_2 - S_0e^r}{S_2 - S_1}$$

Can think of q as a probability since $S_1 < S_0e^r, S_2$ so $0 < q < 1$.

(e) Suppose $V_2 \neq V_1$. Then $V_0^p = e^{-r}[pV_2 + (1-p)V_1] \neq 0$, for if $p > q$, $V_0^p = V_0$

$$\begin{aligned}
\Rightarrow pV_2 + (1-p)V_1 &= V_2 + (1-q)V_1 \\
\Rightarrow p - qV_2 &= (p-q)V_1 \\
\Rightarrow V_2 &= V_1 \text{ if } \ \text{p > q. (indeed if p} \neq \text{q)}
\end{aligned}$$

Thus $V_0^p \neq V_0$, so a trader selling at V_0^p can be arbitraged by either selling at V_0^p if $V_0^p > V_0$ and carrying out risk free strategy (b) or (c) or buying at V_0^p if $V_0^p < V_0$ and shorting the risk free strategy (b) or (c).