Question

Write each of the following second order differential equation as a pair of first order equations for $x(t)$ and $v(t)$ $\Big($where $v = dx/dt\Big)$. Hence obtain a differential equation for v as a function of x. Obtain the general solution of this equation and hence roughly sketch the phase-plane.

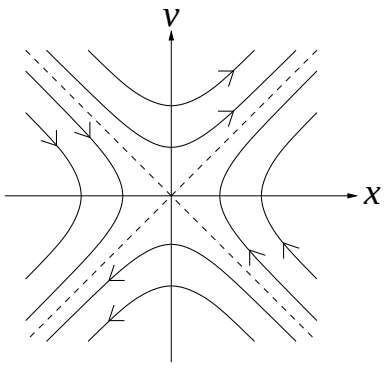
1. $\dfrac{d^2x}{dt^2} - x = 0$
2. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = 0$
3. $\dfrac{d^2x}{dt^2} - x^2 = 0 \qquad (*)$
4. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - \dfrac{dx}{dt}e^x = 0 \qquad (*)$

Answer

a) $\dfrac{dx}{dt} = v, \quad \dfrac{dv}{dt} = x \Rightarrow \dfrac{dx}{dt}\dfrac{dt}{dv} = \dfrac{v}{x} \Rightarrow \dfrac{dx}{dv} = \dfrac{v}{x}$

This is separable $\displaystyle\int x dx = \int v dv \Rightarrow \frac{1}{2}x^2 = \frac{1}{2}v^2 + A$

$\Rightarrow x^2 - v^2 = 2A$, this is a hyperbola.



phase-plane

b) $\dfrac{dx}{dt} = v \Rightarrow \dfrac{dv}{dt} - 2v + x = 0$

$\dfrac{dv}{dt} = 2v - x, \quad \dfrac{dx}{dt} = v$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{2v - x}{v} = 2 - \dfrac{x}{v}$

This is homogeneous, so $v = xV \Rightarrow x\dfrac{dV}{dx} + V = 2 - \dfrac{1}{v}$

$x\dfrac{dV}{dx} = -V + 2 - \dfrac{1}{V}$, this is separable.

$\displaystyle\int \frac{-V dV}{V^2 - 2V + 1} = \int \frac{1}{x} dx$

$\displaystyle\int \frac{-\frac{1}{2}(2V - 2)}{V^2 - 2V + 1} - \frac{1}{(V-1)^2 dV} = \int \frac{1}{x} dx$

$-\dfrac{1}{2}\ln\left(V^2 - 2V + 1\right) + \dfrac{1}{V - 1} = \ln x + A$

$\dfrac{e^{\frac{1}{V-1}}}{\sqrt{V^2 - 2V + 1}} = cx \Rightarrow \dfrac{e^{\frac{x}{v-x}}}{\sqrt{\left(\frac{v}{x}\right)^2 - \frac{2v}{x} + 1}} = cx$

For the phase plane use isoclines / direction fields etc.

DIAGRAM

c) $\dfrac{dx}{dt} = v \Rightarrow \dfrac{dv}{dt} - x^2 = 0$

$\dfrac{dv}{dt} = x^2, \quad \dfrac{dx}{dt} = v \Rightarrow \dfrac{dv}{dx} = \dfrac{x^2}{v}$, this is separable.

$\displaystyle\int v dv = \int x^2 dx$

$\Rightarrow \dfrac{1}{2}v^2 = \dfrac{1}{3}x^3 + A$

Can draw phase plane using isoclines and direction fields.

DIAGRAM

d) $\dfrac{dx}{dt} = v \Rightarrow \dfrac{dv}{dt} + v - ve^x = 0$

$\dfrac{dv}{dt} = -v + ve^x, \quad \dfrac{dx}{dt} = v \Rightarrow \dfrac{dv}{dx} = -1 + e^x$

$\Rightarrow v = -x + e^x + B$

Can draw phase plane by graph or isoclines and direction field.

DIAGRAM