

QUESTION

- (a) State Burnside's formula, carefully defining the terms used in the formula.
- (b) Describe the elements of the rotation group of the cube, giving the order of each element, its fixed set, and describing the orbits of the faces for each rotation.
- (c) Use your answers to parts (a) and (b) to find the number of distinct ways there are to label the faces of a cube with six colours, where each colour may be used more than once. (As usual, "distinct" means that the labellings can be distinguished up to a rotation of the cube, so you will need to consider the action of the rotation group of the cube on the set of all possible labellings.)

ANSWER

- (a) Let G be a finite group acting on a finite set X . For each $g \in G$ let X_g denote the set $\{x \in X | g(x) = x\}$, then the number r of orbits of the action is given by $\left(\sum_{g \in G} |X_g| \right) / |G|$.

- (b) Rotation of order 3 about a diagonal preserving two opposite vertices- 2 orbits each of three mutually adjacent faces.

Rotation of order 2 about a line bisecting opposite edges-3 orbits, each fixed point yields an orbit consisting of two adjacent faces containing that fixed point, and the remaining two faces form the third orbit.

Rotation of order 4 about a line joining the midpoints of opposite faces- each face containing a fixed point is invariant so we obtain two orbits of length one. The remaining four faces split into two orbits of length 2, each consisting of a pair of opposite faces.

The identity element- 6 orbits each of length 1.

- (c) $r = \left(\sum_{g \in G} |X_g| \right) / 24$. There are 8 rotations of the first type and each preserves 6^2 labellings. There are 6 rotations of the second type and each preserves 6^3 labellings. There are 6 rotations of the third type and each preserves 6^3 labellings, and there are 3 rotations of the fourth type each preserving 6^4 labellings. The identity element preserves all 6^6

labellings. Hence there are $\frac{(8.36 + 6.192 + 6.192 + 3.1296 + 46.656)}{12} =$
4428 distinguishable labellings.