Exam Question
Topic: Double Integral in Polars
Evaluate the double integral

$$\iint_R \ln\left(1+x^2+y^2\right)\, d(x,y),$$

where R is the region given by

$$\left\{(x,y)\colon x^2+y^2 \leq 1 \ \text{ and } \ x \leq 0\right\}.$$

Given your answer both in terms of ln and also as a decimal correct to 3 places.

Solution
Changing to polar coordinates gives

$$I = \int_0^1 dr \int_{\pi/2}^{3\pi/2} \ln(1+r^2).r\, d\theta = \pi \int_0^1 r\ln(1+r^2)\, dr.$$

Let $1+r^2 = u$; $\quad 2rdr = du$

$$\begin{aligned} \text{So } \ I &= \pi \int_1^2 \ln u\, du = \frac{\pi}{2}\left[u\ln u - u\right]_1^2 \\ &= \frac{\pi}{2}[2\ln 2 - 2 - 1\ln 1 + 1] = \frac{\pi}{2}[2\ln 2 - 1] = 0.607 \ \ (3 \text{ d.p.}) \end{aligned}$$