Exam Question
Topic: Double Integral
R is the region in the positive quadrant in the x-y plane bounded by

$$x = 0, \ y = x, \ x^2 + y^2 = 1, \ x^2 + y^2 = 0.$$

Show that

$$\iint_R \frac{\ln(x^2 + y^2)}{y} \, d(x, y) = -2\ln(\sqrt{2} - 1)(\ln 27 - 2).$$

[You may use the result that $\tan(\pi/8) = \sqrt{2} - 1$.]

Solution
In polar coordinates

$$\begin{aligned}
I &= \int_{\pi/4}^{\pi/2} d\theta \int_1^3 \frac{\ln(r^2)}{r\sin\theta} r \, dr \\
&= \int_{\pi/4}^{\pi/2} cosec\theta \, d\theta \int_1^3 2\ln r \, dr = \left[\ln\tan\left(\frac{\theta}{2}\right)\right]_{\pi/4}^{\pi/2} 2\left[r\ln r - r\right]_1^3 \\
&= \left[\ln\tan\frac{\pi}{4} - \ln\tan\frac{\pi}{8}\right] 2[3\ln 3 - 3 - \ln 1 + 1] \\
&= [\ln 1 - \ln(\sqrt{2} - 1)] 2[3\ln 3 - 2] = -2\ln(\sqrt{2} - 1)(\ln 27 - 2)
\end{aligned}$$