Question

Solve the following simultaneous equations and identify the intersection points of the corresponding planes

(i)

$$\begin{aligned} 3x + 4y + 2z &= 4 \\ x - 2y - 4z &= 2 \\ 2x + 5y + 3z &= -1 \end{aligned}$$

(ii)

$$\begin{aligned} 4x + 4y + 2z &= 3 \\ 2x - 2y - 4z &= 1 \\ -x + 5y + 3z &= 2 \end{aligned}$$

Answer

(i) From lecture notes:

$$\left.\begin{aligned} 3x + 4y + 2z &= 4 \\ x - 2y - 4z &= 2 \\ 2x + 5y + 3z &= -1 \end{aligned}\right\} cf. \left\{\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right.$$

Thus in short,

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 2 & 4 \\ 1 & -2 & -4 & 2 \\ 2 & 5 & 3 & -1 \end{pmatrix}$$

i.e., $a_1 = 3,\ b_2 = -2$ etc.

Now equations $(4.15a, b, c)$ gives the x solution as:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Calculate

$$\begin{aligned}
\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} &= \begin{vmatrix} 4 & 4 & 2 \\ 2 & -2 & -4 \\ -1 & 5 & 3 \end{vmatrix} \\
&= 4\begin{vmatrix} -2 & -4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -1 & 5 \end{vmatrix} \\
&= 4(-6+20) - 4(6-4) + 2(10-2) \\
&= 56 - 8 + 16 \\
&= \underline{64}
\end{aligned}$$

and

$$\begin{aligned}
\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= \begin{vmatrix} 3 & 4 & 2 \\ 1 & -2 & -4 \\ 2 & 5 & 3 \end{vmatrix} \\
&= 3\begin{vmatrix} -2 & -4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix} \\
&= 3(-6+20) - 4(3+8) + 2(5+4) \\
&= 42 - 44 + 18 \\
&= \underline{16}
\end{aligned}$$

Thus $x = \dfrac{64}{16} = \underline{4}$

Similarly from notes we have:

$$y = \frac{\begin{vmatrix} d_1 & c_1 & a_1 \\ d_2 & c_2 & a_2 \\ d_3 & c_3 & a_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = 16 \text{ from above}$$

$$\begin{vmatrix} d_1 & c_1 & a_1 \\ d_2 & c_2 & a_2 \\ d_3 & c_3 & a_3 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 3 \\ 2 & -4 & 1 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix}$$

$$= 4(-8-3) - 2(4+1) + 3(6-4)$$

$$= -44 - 10 + 6$$

$$= \underline{-48}$$

Thus $y = \dfrac{-48}{16} = \underline{-3}$

Similarly we have

$$z = \frac{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = 16 \text{ from above}$$

$$\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 4 \\ 2 & 1 & -2 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ -1 & 5 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 4(5+4) - 3(10-2) + 4(4+1)$$

$$= 36 - 24 + 20$$

$$= \underline{32}$$

Thus $z = \dfrac{32}{16} = \underline{2}$

Thus $x = 4,\ y = -3,\ z = 2$

are the solutions of this set of simultaneous equations. Hence the 3-planes represented by these 3 equations intersect at a point, P.

PICTURE

(ii)

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 2 & 3 \\ 3 & -2 & -4 & 1 \\ -1 & 5 & 3 & 2 \end{pmatrix}$$

Thus

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Calculate

$$\begin{aligned} \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} &= \begin{vmatrix} 3 & 4 & 2 \\ 1 & -2 & -4 \\ 2 & 5 & 3 \end{vmatrix} \\ &= 3\begin{vmatrix} -2 & -4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix} \\ &= 3(-6+20) - 4(3+8) + 2(5+4) \\ &= 42 - 44 + 18 \\ &= \underline{16} \end{aligned}$$

and

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= \begin{vmatrix} 4 & 4 & 2 \\ 2 & -2 & -4 \\ -1 & 5 & 3 \end{vmatrix} \\ &= 4\begin{vmatrix} -2 & -4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -1 & 5 \end{vmatrix} \\ &= 4(-6+20) - 4(6-4) + 2(10-2) \\ &= 56 - 8 + 16 \\ &= \underline{64} \end{aligned}$$

Thus $x = \frac{16}{64} = \underline{\frac{1}{4}}$

Now also

$$y = \frac{\begin{vmatrix} d_1 & c_1 & a_1 \\ d_2 & c_2 & a_2 \\ d_3 & c_3 & a_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = 64 \text{ from above}$$

$$\begin{aligned}
\begin{vmatrix} d_1 & c_1 & a_1 \\ d_2 & c_2 & a_2 \\ d_3 & c_3 & a_3 \end{vmatrix} &= \begin{vmatrix} 3 & 2 & 4 \\ 1 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix} \\
&= 3\begin{vmatrix} -4 & 2 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} \\
&= 3(+4-6) - 2(-1-4) + 4(3+8) \\
&= -6 + 10 + 44 \\
&= \underline{48}
\end{aligned}$$

Thus $y = \frac{48}{64} = \underline{\frac{3}{4}}$

and finally,

$$z = \frac{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = 64 \text{ from above}$$

$$\begin{aligned}
\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix} &= \begin{vmatrix} 3 & 4 & 4 \\ 1 & 2 & -2 \\ 2 & -1 & 5 \end{vmatrix} \\
&= 3\begin{vmatrix} 2 & -2 \\ -1 & 5 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\
&= 3(10-2) - 4(5+4) + 4(-1-4) \\
&= 24 - 36 - 20 \\
&= \underline{-32}
\end{aligned}$$

Hence $z = \frac{-32}{64} = \underline{-\frac{1}{2}}$

Thus $x = \frac{1}{4},\ y = \frac{3}{4},\ z = \frac{-1}{2}$
are the solutions of this set of simultaneous equations. Hence the 3-planes represented by these 3 equations intersect at a point.