QUESTION (More difficult)

(a) Let $w = \tanh z$. Show that the points of the z-plane on the line $x = a(\neq 0)$ correspond to the points of the circle with centre $(\coth 2a, 0)$ and the radius cosech $2a$ of the w-plane, Indicate in a diagram of the complex plane the way these circles vary with a.

(b) Find all solutions of $\tanh z = 2$.

ANSWER

(a) $w =$tanh$z = \frac{e^z-e^{-z}}{e^z+e^{-z}}$. Let $u = e^z$. Then the line $x = a$ in the u-plane is mapped to points of the form $e^a(\cos y + i\sin y)$, that is to a circle center 0 radius e^a, and u^2 lies on a circle center 0 radius e^2a. Now $w = \frac{u-\frac{1}{u}}{u+\frac{1}{u}} = \frac{u^2-1}{u^2+1}$. This gives $u^2 = \frac{1+w}{1-w}$. Thus $|\frac{1+w}{1-w}| = e^{2a}$, Hence $|1+w|^2 = e^{4a}|1-w|^2$ and so $(1+w)(1+\overline{w}) = e^{4a}(1-w)(1-\overline{w})$. Put $w = X + iY$. Then we find $Y^2 + Y^2 - 2X(\frac{e^{4a}-1}{e^{4a}+1}) + 1 = 0$ which after a bit of work we see is a circle centre (coth $2a$,0), radius cosech $2a$.

(b) Go back to the equation $u^2 = \frac{1+w}{1-w}$. Here $w = 2$, so $u^2 = -3$ and $u = \sqrt{3}i$. Thus $z = \log(\sqrt{3}i) = \log\sqrt{3} + i\frac{\pi}{2} + 2\pi ni = \frac{\log 3}{2} + i(\frac{\pi}{2} + 2\pi n)$