Vector Fields
Conservative Fields

Question
A source of strength 2 is placed at $(0,0,0)$ and a sink of strength 1 is placed at $(0,0,1)$. Find the velocity field for this system. Show that the velocity is vertical at all points of a certain sphere. Sketch the streamlines of the flow.
Answer
The scalar potential of the system is

$$\phi(x,y,z) = \phi(\underline{R}) = -\frac{2}{|\underline{r}|} + \frac{1}{|\underline{r}-\underline{k}|}.$$

This gives the velocity field

$$\begin{aligned}\underline{v} &= \nabla\phi = \frac{2\underline{r}}{|\underline{r}|^3} - \frac{\underline{r}-\underline{k}}{|\underline{r}-\underline{k}|^3} \\ &= \frac{2(x\underline{i}+y\underline{j}+z\underline{k})}{(x^2+y^2+z^2)^{3/2}} - \frac{x\underline{i}+y\underline{j}+(z-1)\underline{k}}{(x^2+y^2+(z-1)^2)^{3/2}}.\end{aligned}$$

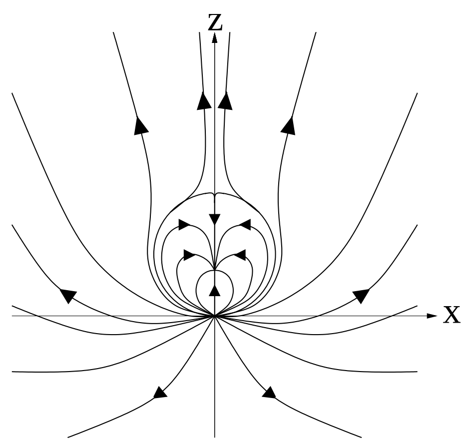
In order to have vertical velocity

$$\frac{2x}{(x^2+y^2+z^2)^{3/2}} = \frac{x}{(x^2+y^2+(z-1^2)^{3/2}}$$

with a similar equation for y. Both of these equation will be satisfied at all points of the z-axis, and where

$$\begin{aligned}2(x^2+y^2+(z-1)^2)^{3/2} &= (x^2+y^2+z^2)^{3/2} \\ 2^{2/3}(x^2+y^2+(z-1)^2) &= x^2+y^2+z^2 \\ x^2+y^2+(z-K)^2 &= K^2-K,\end{aligned}$$

with $K = 2^{2/3}/(2^{2/3}-1)$.
The latter equation represents a sphere S, as $K^2-K>0$. The velocity is vertical on all point of S, as well as at all points of the z-axis.
As the source (at $(0,0,0)$) is twice as strong as the sink (at $(0,0,1)$), half of the fluid that the sources emits will go to the sink. By a process of symmetry, this will be the upper half plane $z>0$. All of the fluid emitted below $z=0$ will flow out to infinity.
There exists one point with $\underline{v}=\underline{0}$. This point $((0,0,2+\sqrt{2}))$ lies inside S. Any streamlines that emerge from the origin parallel to the xy-plane lead to this point. Any streamlines that emerge into $z>0$ cross S and approach the sink. Any streamlines that emerge into $z<0$ flow to infinity.
Some cross S twice, some are tangent to S and some do not intersect S at all.


z
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