Vector Fields
Conservative Fields

Question
Two sources of strength m are placed at $(0, 0, \pm l)$. Find the velocity due to these sources, and state where the velocity is zero. Determine the velocity at the point $(x, y, 0)$ in the xy-plane, and state where in the xy-plane the speed is greatest.
Answer
For the two-source system

$$\phi(x,y,z) = \phi(\underline{r}) = -\frac{m}{|\underline{r} - l\underline{k}|} - \frac{m}{|\underline{r} + l\underline{k}|}.$$

This gives the velocity field

$$\begin{aligned} \underline{v}(\underline{r}) &= \nabla\phi(\underline{r}) \\ &= \frac{m(\underline{r} - l\underline{k})}{|\underline{r} - l\underline{k}|^3} + \frac{m(\underline{r} + l\underline{k})}{|\underline{r} + l\underline{k}|^3} \\ &= \frac{m(x\underline{i} + y\underline{j} + (z-l)\underline{k})}{[x^2+y^2+(z-l)^2]^{3/2}]} + \frac{m(x\underline{i} + y\underline{j} + (z+l)\underline{k})}{[x^2+y^2+(z+l)^2]^{3/2}}. \end{aligned}$$

Notice that $v_1 = 0$ if and only if $x = 0$, and $v_2 = 0$ if and only if $y = 0$.

$$\underline{v}(0,0,z) = m\left(\frac{z-l}{|z-l|^3} + \frac{z+l}{|z+l|^3}\right)\underline{k}$$

is only $\underline{0}$ if and only if $z = 0$. So $\underline{v} = \underline{0}$ at the origin only.
For points in the xy-plane

$$\underline{v}(x,y,0) = \frac{2m(x\underline{i} + y\underline{j})}{(x^2+y^2+l^2)^{3/2}}.$$

So the velocity is radially away from the origin in the plane, as is required by symmetry. The speed at $(x, y, 0)$ is given by

$$\begin{aligned} v(x,y,0) &= \frac{2m\sqrt{x^2+y^2}}{(x^2+y^2+l^2)^{3/2}} \\ &= \frac{2ms}{(s^2+l^2)^{3/2}} = g(s) \\ \text{with } s &= \sqrt{x^2+y^2} \end{aligned}$$

For max $g(s)$

$$\begin{aligned} 0 &= g'(s) = 2m\frac{(s^2+l^2)^{3/2} - \frac{3}{2}s(s^2+l^2)^{1/2}2s}{(s^2+l^2)^3} \\ &= \frac{2m(l^2-2s^2)^{3/2}}{(s^2+l^2)^{5/2}}. \end{aligned}$$

So the speed in the xy plane is at its greatest when it the points of the circle $x^2+y^2 = l^2/2$.