Vector Fields
Conservative Fields

Question
For the following vector field, find whether it is conservative. If so, find a corresponding potential
$\underline{F}(x,y,z) = (2xy - z^2)\underline{i} + (2yz + x^2)\underline{j} - (2zx - y^2)\underline{k}$
Answer

$$\begin{aligned}
\Rightarrow F_1 &= 2xy - z^2 \\
F_2 &= 2yz + x^2 \\
F_3 &= y^2 - 2zx \\
\Rightarrow \frac{\partial F_1}{\partial y} &= 2x = \frac{\partial F_2}{\partial x} \\
\frac{\partial F_1}{\partial z} &= -2z\frac{\partial F_3}{\partial x} \\
\frac{\partial F_2}{\partial z} &= 2y = \frac{\partial F_3}{\partial y}
\end{aligned}$$

$\Rightarrow \underline{F}$ can be conservative
If $\underline{F} = \nabla\phi$

$$\begin{aligned}
\Rightarrow \frac{\partial \phi}{\partial x} &= 2xy - z^2 \\
\frac{\partial \phi}{\partial y} &= 2yz + x^2 \\
\frac{\partial \phi}{\partial z} &= y^2 - 2zx \\
\Rightarrow \phi(x,y,z) &= \int (2xy - z^2)\,dx \\
&= x^2y - xz^2 + C_1(y,z) \\
2yx + x^2 &= \frac{\partial \phi}{\partial y} = x^2 + \frac{\partial C_1}{\partial y} \\
\Rightarrow \frac{\partial C_1}{\partial y} &= 2yz \\
\Rightarrow C_1(y,z) &= y^2z + C_2(z) \\
\\
\phi(x,y,z) &= x^y - xz^2 + y^2z + C_2(z) \\
y^2 - 2zx &= \frac{\partial \phi}{\partial z} = -2xz + y^2 + C_2'(z) \\
\Rightarrow C_2'(z) &= 0.
\end{aligned}$$

So $\phi(x,y,z) = x^2y - xz^2 + y^2z$ is a scalar potential for $\underline{F}$, and $\underline{F}$ is conservative on $\Re^3$.