Vector Fields
Conservative Fields

Question
The vector field $\underline{F}$ is given by

$$\underline{F}(x,y,z) = \frac{2x}{z}\underline{i} + \frac{2y}{z}\underline{j} - \frac{x^2+y^2}{z^2}\underline{k}.$$

Show that $\underline{F}$ is conservative, and find the potential. Describe the equipotential surfaces and find the field lines of $\underline{F}$.
Answer

$$\begin{aligned} F_1 &= \frac{2x}{z} \\ F_2 &= \frac{2y}{z} \\ F_3 &= -\frac{x^2+y^2}{z^2} \end{aligned}$$

This gives

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= 0 = \frac{\partial F_2}{\partial x} \\ \frac{\partial F_1}{\partial z} &= -\frac{2x}{z^2} = \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial z} &= -\frac{2y}{z^2} = \frac{\partial F_3}{\partial y}. \end{aligned}$$

$\Rightarrow \underline{F}$ can be conservative in $\Re^3$ except on the plane $z = 0$ where it is not defined. If $\underline{F} = \nabla\phi$

$$\begin{aligned} \Rightarrow \frac{\partial \phi}{\partial x} &= \frac{2x}{z} \\ \frac{\partial \phi}{\partial y} &= \frac{2y}{z} \\ \frac{\partial \phi}{\partial z} &= -\frac{x^2+y^2}{z^2} \\ \Rightarrow \phi(x,y,z) &= \int \frac{2x}{z}\,dx \\ &= \frac{x^2}{z} + C_1(y,z) \end{aligned}$$

$$\frac{2y}{z} = \frac{\partial \phi}{\partial y} = \frac{\partial C_1}{\partial y}$$

$$\begin{aligned}
\Rightarrow C_1(y,z) &= \frac{y^2}{z} + C_2(z) \\
\phi(x,y,z) &= \frac{x^2+y^2}{z} + C_2(z) \\
-\frac{x^2+y^2}{z^2} &= \frac{\partial \phi}{\partial z} = -\frac{x^2+y^2}{z^2} + C_2'(z) \\
\Rightarrow C_2(z) &= 0
\end{aligned}$$

So $\phi(x,y,z) = \dfrac{x^2+y^2}{z}$ is a potential for $\underline{F}$, and $\underline{F}$ is conservative on $\Re^3$, except where it is not defined on $(z = 0)$.

The equipotential surfaces will have the equations $\dfrac{x^2+y^2}{z} = C$ or $Cz = x^2 + y^2$. Therefore the surfaces are circular paraboloids.

The field lines of $\underline{F}$ satisfy

$$\frac{dx}{\frac{2x}{z}} = \frac{dy}{\frac{2y}{z}} = \frac{dz}{-\frac{x^2+y^2}{z^2}}$$

So it can be seen that $\frac{dx}{x} = \frac{dy}{y}$, $\Rightarrow y = Ax$ for an arbitrary constant A.

$$\begin{aligned}
\Rightarrow \frac{dx}{2x} &= \frac{z\,dz}{-(x^2+y^2)} \\
&= \frac{z\,dz}{-x^2(1+A^2)} \\
\Rightarrow -(1+a^2)x\,dx &= 2z\,dz.
\end{aligned}$$

And so

$$\frac{1+A^2}{2}x^2 + z^2 = \frac{B}{2}$$

or

$$x^2 + y^2 + 2z^2 = B$$

with B being a second arbitrary constant. So the field lines of $\underline{F}$ are the ellipses in which the vertical planes containing the z-axis intersects the ellipsoids $x^2 + y^2 + 2z^2 = B$. These are orthogonal to all the equipotential surfaces of $\underline{F}$.