QUESTION
Evaluate the following real Fourier integrals:

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos nx}{x^2+x+1}\, dx$

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin nx}{x^2+x+1}\, dx$

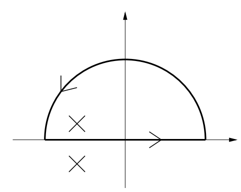
(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2(x^2+2)}\, dx$

Hint: to solve $\int \cos x f(x)\, dx$, where $f(x)$ is a real function of x, solve $\int e^{ix} f(x)\, dx$, then take the real part of the result.
Similarly, to calculate $\int \sin x f(x)\, dx$, take the imaginary part. Remember that to evaluate $\int e^{ix} f(x)\, dx$, you must close the contour with a semicircle in the upper half- if you closed in the lower half the semicircle would contribute.

ANSWER

(i) $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos(nx)}{x^2+x+1} = \text{Re}J$ where $J = \displaystyle\int_{-\infty}^{\infty} \frac{e^{inx}}{x^2+x+1}$

Closed contour in upper half plane



Simple poles at $z^2+z+1=0$, $(z+\frac{1}{2})^2 = \frac{1}{4}-1$, $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$; only $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}$ is inside the closed contour.

$$\begin{aligned}
\text{J} &= 2\pi i \text{Res}\left(\frac{e^{inz}}{z^2+z+1}, -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\
&= 2\pi i \lim_{z\to -\frac{1}{2}+\frac{\sqrt{3}}{2}i} \frac{e^{inz}}{2z+1} \\
&= 2\pi i \frac{e^{in\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}}{\sqrt{3}i} \\
&= \frac{2\pi}{\sqrt{3}} e^{-\frac{1}{2}in-\frac{\sqrt{3}}{2}n}
\end{aligned}$$

$I = \text{Re}J = \frac{2\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}n} \cos\frac{n}{2}$

(ii)

$$\int_{-\infty}^{\infty} \frac{\sin(nx)}{x^2+x+1} = \mathrm{Im}J = -\frac{2\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}n} \sin\frac{n}{2}$$

(iii) $I = \mathrm{Re}J, J = \int_\gamma \frac{e^{iz}}{(z^2+1)^2(z^2+2)}$

Poles inside the contour are : $z = i$ a double pole and $z = \sqrt{2}i$ a simple pole.

$$\mathrm{Res}\left(\sqrt{2}i\right) = \frac{e^{i\left(\sqrt{2}i\right)}}{\left(\left(\sqrt{2}\right)^2+1\right)^2 2\left(\sqrt{2}i\right)} = \frac{e^{-\sqrt{2}}}{2\sqrt{2}i}$$

$$\mathrm{Res}(i) = \lim_{z\to i} \frac{d}{dz} \frac{e^{iz}}{(z^2+2)(z+i)^2}$$

$$= \lim_{z\to i} \left(\frac{ie^{iz}}{(z^2+2)(z+i)^2} - \frac{2ze^{iz}}{(z^2+2)^2(z+i)^2} \right) - \left(\frac{2e^{iz}}{(z^2+2)(z+i)^3} \right)$$

$$= e^{-1} \left(\frac{1}{1.(2i)^2} - \frac{2i}{1^2(2i)^2} - \frac{2}{1(2i)^3} \right) = e^{-1} \left(-\frac{i}{4} + \frac{i}{2} - \frac{i}{4} \right) = 0$$

$J = 2\pi i \frac{e^{-\sqrt{2}}}{2\sqrt{2}i} = \frac{\pi}{\sqrt{2}} e^{-\sqrt{2}}$. $I = J$