QUESTION

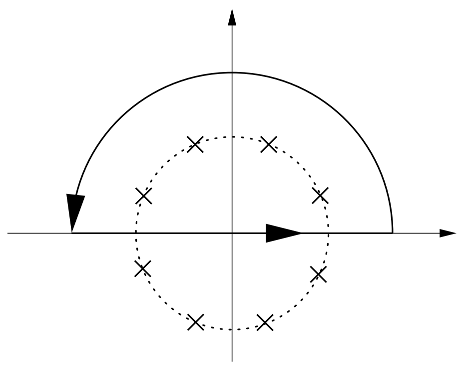
Evaluate the following real integrals by using the residue theorem:

$$\text{(i)} \int_{-\infty}^{\infty} \frac{dx}{1+x^8} \quad \text{(ii)} \quad \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \quad \text{(iii)} \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(x^2+9)}$$

You can close the contour either in the lower or upper half here, so take the upper half.

ANSWER

(i) $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^8} = \int_C \frac{dz}{1+z^4}$



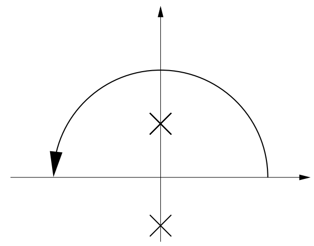
The integration contour encloses simple poles at

$$z_0 = e^{\frac{i\pi}{8}},\ e^{\frac{3i\pi}{8}},\ e^{\frac{5i\pi}{8}},\ e^{\frac{7i\pi}{8}}$$

$$\text{Res}\left(\frac{1}{1+z^8}, z_0\right) = \frac{1}{8z_0^7} = \frac{z_0}{8z_0^8} = -\frac{1}{8}z_0$$

$$\begin{aligned} I &= 2\pi i\left(-\frac{1}{8}\right)\left(e^{\frac{i\pi}{8}} + e^{\frac{3i\pi}{8}} + +e^{\frac{5i\pi}{8}} + e^{\frac{7i\pi}{8}}\right) \\ &= -\frac{\pi i}{4}\left(e^{i\pi}8 + e^{\frac{3i\pi}{8}} - e^{-\frac{3i\pi}{8}} - e^{-\frac{i\pi}{8}}\right) \\ &= \frac{\pi}{2}\left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8}\right) \end{aligned}$$

(ii) $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_\gamma \frac{dz}{(z+i)^2(z-i)^2}$ has double poles at $z = \pm i$, but only $z = i$ is inside the contour.



$$\begin{aligned}
\text{Res}\left(\frac{1}{(1+z^2)}, i\right) &= \lim_{z\to i}\frac{d}{dz}\frac{1}{(z+i)^2} \\
&= \lim_{z\to i} -\frac{2}{(z+i)^3} = -\frac{i}{4} \\
I &= 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}
\end{aligned}$$

(iii) $I = \int \frac{dx}{(x^2+4)(x^2+9)}$ has simple poles inside the contour at $z = 2i$ and $3i$.

$$\begin{aligned}
\text{Res}(2i) &= \frac{1}{(2i+2i)(2i+3i)(2i-3i)} = -\frac{i}{20} \\
\text{Res}(3i) &= \frac{1}{(3i+3i)(3i+2i)(3i-2i)} = -\frac{i}{30} \\
I &= 2\pi i\left(-\frac{i}{20} - \frac{i}{30}\right) = \frac{\pi}{30}
\end{aligned}$$