QUESTION

If C is the unit circle centred on the origin, evaluate by the residue theorem

$$\text{(i)} \int_C \frac{\cos z}{2z^2 - 7z}\, dz \quad \text{(ii)} \quad \int_C \frac{2z+1}{2z^4 - z^3}\, dz \quad \text{(iii)} \quad \int_C \frac{e^z}{8z^3 - 1}\, dz$$

ANSWER

(i) $\displaystyle\int_{|z|=1} \frac{\cos z}{2z62 - 7z}\, dz$ has simple poles at $z = 0$ and $z = \frac{7}{2}$. Hence

$$\int_{|z|=1} \frac{\cos z}{2z^2 - 7z}\, dz = 2\pi i \text{Res}(0) = 2\pi i \lim_{z\to 0} \frac{\cos z}{4z - 7} = -\frac{2\pi i}{7}.$$

(ii) $\displaystyle\int_{|z|=1} \frac{2z+1}{2z^4 - z^3}\, dz$

has a simple pole at $z = \frac{1}{2}$ and a triple pole at $z = 0$.

$$\text{Res}\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{2z+1}{8z^3 - 3z} = 8$$

$$\text{Res}(0) = \frac{1}{2!} \lim_{z\to\frac{1}{2}} \left(\frac{d}{dz}\right)^2 \frac{2z+1}{2z-1} = \ldots$$

Easier:

$$\begin{aligned} \frac{1}{z^3}\frac{2z+1}{2z-1} &= -\frac{1}{z^3}(2z+1)\frac{1}{1-2z}\text{(use geometric series)} \\ &= -\frac{1}{z^2}(1+2z)(1+2z+4z^2+\ldots) \\ &= -\frac{1}{z^3}(1+4z+8z^2+\ldots) \\ &\Rightarrow \text{Res}(0) = -8 \Rightarrow I = 0 \end{aligned}$$

(iii) $I = \displaystyle\int_{z=0} \frac{e^z}{8^3 - 1}\, dz$ has simple poles at $z = \frac{1}{2},\ \frac{1}{2}e^{\frac{2\pi i}{3}},\ \frac{1}{2}e^{\frac{4\pi i}{3}}$

$$\text{Res}\left(\frac{e^z}{8z^3-1}, z_0\right) = \lim_{z\to z_0} \frac{e^z}{24z^2} = \frac{e^{z_0}}{24z_0} = \frac{z_0 e^{z_0}}{24z_0^3} = \frac{z_0 e^{z_0}}{3}$$

$$\begin{aligned} I &= 2\pi i \frac{1}{3}\left(\frac{1}{2}e^{\frac{1}{2}} + \frac{1}{2}e^{\frac{2\pi i}{3}} e^{-\frac{1}{4}+\frac{\sqrt{3}}{4}i} + \frac{1}{2}e^{\frac{4\pi i}{3}} e^{-\frac{1}{4}-\frac{\sqrt{3}}{4}i}\right) \\ &= \frac{\pi i}{3}\left[e^{\frac{1}{2}} + e^{-\frac{1}{4}} 2\cos\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right)\right] \end{aligned}$$