QUESTION

(i) Using partial fractions and the table of inverse Laplace transforms show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+1)^2}\right\} = e^{-2t} - e^{-t} + te^{-t}.$$

(ii) Use Laplace transforms and part (i) to find the solution of

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 2e^{-t}$$

which satisfies the conditions $x = 0$ and $\dfrac{dx}{dt} = 0$ when $t = 0$.

What is the behaviour of the solution x and $\dfrac{dx}{dt}$ as $t \to \infty$?

ANSWER

(i)

$$\begin{aligned}\frac{1}{(s+2)(s+1)^2} &= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \\ &= \frac{A(s+1)^2 + B(s+1)(s+2) + C(s+2)}{(s+2)(s+1)^2}\end{aligned}$$

therefore $A(s+1)^2 + B(s+1)(s+2) + C(s+2) = 1$

$s = -1$	$0 + 0 + C(1) = 1,$	$C = 1$
$s = -2$	$A(-1)^2 + 0 + 0 = 1$	$A = 1$
coefficient of s^2	$A + B = 0,$	$B = -A = -1$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}\right\} = e^{-2t} - e^{-t} + te^{-t}$$

(ii) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = 2e^{-t}, \qquad x(0) = \dfrac{dx}{dt}(0) = 0$

Taking the Laplace transform

$$
\begin{aligned}
\left(s^2X - sx(0) - \frac{dx}{dt}(0)\right) + 3(sX - x(0)) + 2X &= \frac{2}{s+1} \\
s^2X + 3sX + 2X &= \frac{2}{s+1} \\
(s^2 + 3s + 2)X &= \frac{2}{s+1} \\
(s+2)(s+1)X &= \frac{2}{s+1} \\
X &= \frac{2}{(s+2)(s+1)^2}
\end{aligned}
$$

Hence $x(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s+2)(s+1)^2}\right\} = 2(e^{-2t} - e^{-t} + te^{-t})$
i.e. $x(t) = 2\left(\frac{1}{e^{2t}} - \frac{1}{e^t} + \frac{t}{e^t}\right) \to 0$ as $t \to \infty$.

$$
\begin{aligned}
\frac{dx}{dt} &= 2\{-2e^{-2t} + e^{-t} + e^{-t} - te^{-t}\} \\
&= 2\{-2e^{-2t} + 2e^{-t} - te^{-t}\} \\
&= 2\left(-\frac{2}{e^{2t}} + \frac{2}{e^t} - \frac{t}{e^t}\right) \\
&\to 0 \text{ as } t \to \infty
\end{aligned}
$$