QUESTION

(a) Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

and determine the corresponding eigenvectors.

(b) Writing $z = x + jy$ find the locus of the point z in the Argand diagram which satisfies the equation $\left|\dfrac{z-2j}{z+1}\right| = 1$.
Illustrate your result on an Argand diagram, and explain briefly how the locus is geometrically related to the points $2j$ and -1.

ANSWER

(a) $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Eigenvalues satisfy $\det(A - \lambda I) = 0$

$$\begin{aligned}
\det(A-\lambda I) &= \det\begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} \\
&= (1-\lambda)^2 - 3^2 \\
&= (1-\lambda-3)(1-\lambda+3) \\
&= (-\lambda-2)(4-\lambda) \\
&= (\lambda-4)(\lambda+2) \\
&= 0 \text{ if } \lambda = 4, -2.
\end{aligned}$$

$\lambda = 4:$
$(A - 4I)\mathbf{X} = 0$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -3x_1 + 3x_2 &= 0 \\ 3x_1 - 3x_2 &= 0 \end{aligned}\right\} \; x_1 = x_2 = C$$

Therefore the eigenvector is $C\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -2:$
$(A - (-2)I)\mathbf{X} = 0, \; (A + 2I)\mathbf{X} = 0$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} 3x_1 + 3x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{array} \right\} \; x_2 = -x_1, \Rightarrow x_1 = D, \; x_2 = -D$$

Therefore the eigenvector is $D \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) $\left| \dfrac{z-2j}{z+1} \right| = 1$, i.e. $\dfrac{|z-2j|}{|z+1|} = 1$, or $|z-2j| = |z+1|$

Using $z = x+jy$, $|x+jy-2j| = |x+jy+1|$, or $|x+j(y-2)| = |x+jy+1|$

i.e.$\{x^2 + (y-2)^2\}^{\frac{1}{2}} = \{(x+1)^2 + y^2\}^{\frac{1}{2}}$

Squaring both sides gives $x^2 + (y-2)^2 = (x+1)^2 + y^2$

i.e. $x^2 + y^2 - 4y + 4 = x^2 + 2x + 1 + y^2, \; 4y = -2x + 3$

i.e. $y = -\dfrac{1}{2}x + \dfrac{3}{4}$.

$|z - 2j|$ = distance of z from $2j$, $|z+1|$ = distance of z from -1.

The distances are the same along the straight line shown.