QUESTION

A line L_1 passes through the two points $A(1,2,-1)$ and $B(2,-3,1)$. A second line L_2, which is parallel to the vector $(0,-2,1)$, passes through the point $C(2,1,-1)$.

(i) Obtain the vector equations of the lines L_1 and L_2 and show that they intersect at the point B.

(ii) Find a vector $\mathbf{n}$ which is perpendicular to L_1 and L_2, and hence obtain the vector equation of the plane P_1 which contains the lines L_1 and L_2 and passes through B.

(iii) Obtain the vector equation of a second plane P_2 which is parallel to P_1 and passes through $D(-2,1,-1)$.

(iv) Find the distance between the planes P_1 and P_2.

ANSWER

$A(1,2,-1),\ B(2,-3,1),\ C(2,1,-1)$

(i) $L_1:\ \vec{AB} = (2-1,-3-2,1-(-1)) = (1,-5,2)$ so the equation of the line is

$\mathbf{r} = (1,2,-1) + s(1,-5,2) = (1+s, 2-5s, -1+2s)$

$L_2:\ \mathbf{r} = (2,1,-1) + t(0,-2,1) = (2, 1-2t, -1+t)$

The lines intersect when $(1+s, 2-5s, -1+2s) = (2, 1-2t, -1+t)$

i.e. $1+s = 2 \Rightarrow s = 1$

$2-5s = 1-2t \Rightarrow 2t = 5s-1 = 5(1)-1 = 4 \Rightarrow t = 2$

[Check : $-1+2s = -1+2 = 1;\ -1+t = -1+2 = 1$]

Therefore the point of intersection is $(1+1, 2-5, -1+2) = (2,-3,1)$

(ii) A vector parallel to L_1 is $(1,-5,2)$

a vector parallel to L_2 is $(0,-2,1)$

Therefore

$$\begin{aligned} \mathbf{n} &= (1,-5,2)\times(0,-2,1) \\ &= (-5(1)-2(-2), 2(0)-1(1), 1(-2)-0(-5)) \\ &= (-5+4, 0-1, -2-0) = (-1,-1,-1) \end{aligned}$$

The equation of P_1 is $\mathbf{r}\cdot\mathbf{n} = C_1$.

A lies on the plane so

$$\begin{aligned} C_1 &= \mathbf{a}\cdot\mathbf{n} = (1,2,-1).(-1,-1,-2) \\ &= 1(-1)+2(-1)+(-1)(-2) = -1-2+2 = -1 \end{aligned}$$

Therefore $\mathbf{r}\cdot(-1,-1,-2) = -1$ or $\mathbf{r}\cdot(1,1,2) = 1$.

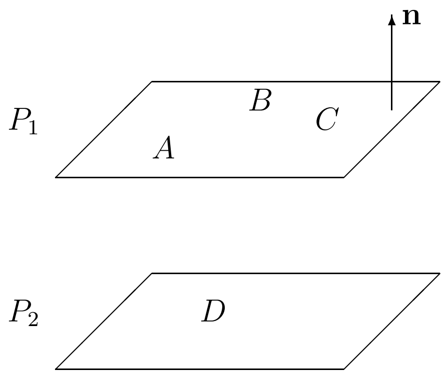
(iii) Parallel planes have the same normal so the equation of the plane is

$\mathbf{r} \cdot (-1, -1, -2) = k$
D is on the plane so

$$\begin{aligned} k &= (-2, 1, -1) \cdot (-1, -1, -2) \\ &= -2(-1) + 1(-1) - 1(-2) = 2 - 1 + 2 = 3 \end{aligned}$$

i.e. $\mathbf{r} \cdot (-1, -1, -2) = 3$ or $\mathbf{r} \cdot (1, 1, 2) = -3$

(iv) There are many ways to obtain this answer, for example



$$\begin{aligned} \text{distance} &= |\vec{DB} \cdot \hat{\mathbf{n}}| \\ &= \left| (2 - (-2), -3 - 1, 1 - (-1)) \cdot \frac{(1, 1, 2)}{\sqrt{1^2 + 1^2 + 2^2}} \right| \\ &= |(4, -4, 2) \cdot (1, 1, 2)| \frac{1}{\sqrt{6}} \\ &= \frac{4 - 4 + 4}{\sqrt{6}} = \frac{4}{\sqrt{6}} \end{aligned}$$