QUESTION

(a) State the Duality Theorem of linear programming and use it to prove the Theorem of Complementary Slackness.

(b) Use duality theory to determine whether $x_1 = 3$, $x_2 = 0$, $x_3 = 0$, $x_4 = 2$, is an optimal solution of the linear programming problem

$$\begin{array}{ll} \text{maximize} & z = 10x_1 - 13x_2 + 25x_3 + x_4 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0, \\ & 4x_1 - 3x_2 + 7x_3 + 2x_4 \leq 16 \\ & 5x_1 + 6x_2 + x_3 + 4x_4 \leq 24 \\ & 2x_1 + 2x_2 - 3x_3 + 5x_4 = 16. \end{array}$$

Does your conclusion remain the same if the objective is changed to

$$\text{maximize } z' = 10x_1 - 12x_2 + 22x_3 + x_4$$

but the constraints are unaltered?

(c) Consider a linear programming problem having constraints of the form

$$\begin{array}{ll} x_j \geq 0 & \text{for } i = 1, \ldots, n \\ \sum_{j=1}^{n} a_{ij} x_j = b_i & \text{for } i = 1, 2. \end{array}$$

An alternative linear programming problem is identical, except that the last constraint is replaced by

$$\sum_{j=1}^{n} (a_{ij} + a_{2j}) x_j = b_1 + b_2$$

i.e., the first equation is added to the second, which yields an equivalent problem. Given that these two linear programming problems have optimal solutions, analyze how the optimal values of the dual variables of the two problems are related.

ANSWER

(a) The duality theorem states that:

- if the primal problem has an optimal solution, then so has the dual, and $z_P = z_D$;

- if the primal problem is unbounded, then the dual is infeasible;
- if the primal problem is infeasible, then the dual is either infeasible or unbounded.

Consider the following primal and dual problems

Maximize	$z_P = \mathbf{c}^T\mathbf{x}$	Minimize	$z_D = \mathbf{b}^T\mathbf{y}$
subject to	$\mathbf{x} \geq 0, \mathbf{s} \geq \mathbf{0}$	subject to	$\mathbf{y} \geq \mathbf{0}, \mathbf{t} \geq \mathbf{0}$
	$A\mathbf{x} + \mathbf{s} = \mathbf{b}$		$A^T\mathbf{y} - \mathbf{t} = \mathbf{c}$

For optimal solutions, complementary slackness states that $y_i s_i = 0$ $(i = 1, \ldots, m)$, $t_j x_j = 0$ $(j = 1, \ldots, n)$.

For feasible solutions of the primal and the dual, we have $c_j = 1, \ldots, n$

$$z_P = \mathbf{c}^T\mathbf{x} = (\mathbf{y}^T A - \mathbf{t}^T)\mathbf{x} = \mathbf{y}^T(\mathbf{b} - \mathbf{s}) - \mathbf{t}^T x = z_D - \mathbf{y}^T\mathbf{s} - \mathbf{t}^T$$

For an optimal solution of the primal and dual, $z_P = z_D$ so

$$\mathbf{y}^T\mathbf{s} + \mathbf{t}^T\mathbf{x} = 0$$

Since variables are non-negative this implies that

$$\begin{aligned} y_i s_i &= 0 \; i = 1, \ldots, m \\ t_j x_j &= 0 \; j = 1, \ldots, n \end{aligned}$$

(b) If s_1, s_2 are slack variables for the first two constraints then $s_1 = 0,\ s_2 = 1$ for the proposed solution.

The dual problem is

minimize	$z_d = 16y_1 + 24y_2 + 16y_3$
subject to	$y_1 \geq 0, y_2 \geq 0$
	$4y_1 + 5y_2 + 2y_3 - t_1 = 10$
	$-3y_1 + 6y_2 + 2y_3 - t_2 = -13$
	$7y_1 + y_2 - 3y_3 - t_3 = 25$

$2y_1 + 4y_2 + 5y_3 - t_4 = 1$

for slack variables $t_1 \geq 0,\ t_2 \geq 0,\ t_3 \geq 0,\ t_4 \geq 0$.

If the proposed solution is optimal, then we can use complementary slackness to obtain

$$t_1 = 0,\ t_4 = 0,\ y_2 = 0$$

The first and last dual constraints become

$$\begin{aligned} 4y_1 + 2y_3 &= 10 \\ 2y_1 + 5y_3 &= 1 \end{aligned}$$

Solving yields $y_1 = 3,\ y_3 = -1$ and we obtain $t_2 = 2, t_3 = -1$.

We require that $t_3 \geq 0$, so the solution is not optimal.

For the modified objective, computations are the same, except that $t_2 = 1, t_3 = 2$, so the dual solution is feasible.

Since $z = 32 = z_D$, both solutions are optimal.

(c) Let the rows of the two original constraints be $\mathbf{a}_1 T,\ \mathbf{a}_2^T$.

The original problem is of the form

$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{a}_1^T\mathbf{x} = b_1 \\ & \mathbf{a}_2^T\mathbf{x} = b_2\ \mathbf{x} \geq 0 \end{array}$$

The dual is

$$\begin{array}{ll} \text{Minimize} & b_1 y_1 + b_2 y_2 \\ \text{subject to} & \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq \mathbf{c} \end{array}$$

The dual of the alternative problem is

$$\begin{array}{ll} \text{Minimize} & b_1 y_1' + (b_1 + b_2) y_2' \\ \text{subject to} & \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_1 + \mathbf{a}_2 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \geq \mathbf{c} \end{array}$$

Clearly the values $y_2' = y_2\ y_1' = y_1 - y_2$ give the same objective function value and left-hand side of the constraints