QUESTION

(a) Consider the linear programming problem

$$\begin{array}{lll} \text{maximize} & \sum\limits_{j=1}^{n} c_j x_j & \\ \text{subject to} & x_j \geq 0 & j = 1, \ldots, n \\ & \sum\limits_{j=1}^{n} a_{ij} x_j = b_i & i = 1, \ldots, m, \end{array}$$

where the constraint matrix $A = (a_{ij})$ has rank m, and $m < n$. Explain briefly what is meant by a *basic feasible solution* of this problem. Prove that an extreme point of the convex set of feasible solutions is a basic feasible solution.

(b) Suppose that there are artificial basic variables at the end of phase 1 of the two-phase simplex method. State the circumstances under which phase 2 should be performed, and explain what modifications are necessary to deal with these artificial variables.

(c) Solve the following linear programming problem with the dual simplex method.

$$\begin{array}{ll} \text{Minimize} & z = 16x_1 + 15x_2 + 4x_3 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\ & 2x_1 + 4x_2 + 5x_3 \geq 25 \\ & 8x_1 + 3x_2 - 4x_3 \geq 28. \end{array}$$

ANSWER

(a) After a renumbering of the variables, a basic feasible solution is a vector $\mathbf{x}^T = (x_1, \ldots, x_m, 0, \ldots, 0)$ where $x_j \geq 0$ for $j = 1, \ldots, m$ that satisfies the constraints. Let $\mathbf{x}_0^T = (x_1^0, \ldots, x_r^0, 0, \ldots, 0)$ be an extreme point in which the first r components are positive, where $r > m$. Let $\mathbf{a}_1, \ldots \mathbf{a}_r$ be the first r columns of A. Since $r > m$, $\mathbf{a}_1, \ldots \mathbf{a}_r$ are linearly dependent, i.e. there exist $\lambda_1, \ldots, \lambda_r$, not all zero such that

$$\lambda \mathbf{a}_1 + \ldots + \lambda_r \mathbf{a}_r = \mathbf{0}$$

Let

$$\epsilon = min \left\{ \frac{x_j^0}{|\lambda_j|} : \lambda_j \neq 0,\ j = 1, \ldots, r \right\}$$

and consider $\mathbf{x}_0 + \epsilon\lambda,\ \mathbf{x}_2 = \mathbf{x}_0 - \epsilon\lambda$, where $\lambda^T = (\lambda_1, \ldots \lambda_r, 0, \ldots, 0)$. The definition of ϵ ensures that $\mathbf{x}_1 \geq \mathbf{0}$ and $\mathbf{x}_2 \geq \mathbf{0}$. Furthermore,

$$A\mathbf{x}_1 = a(\mathbf{x}|_0 + \epsilon\lambda) = \mathbf{b}\ A\mathbf{x}_2 = A(\mathbf{x}_0 - \epsilon\lambda) = \mathbf{b}$$

so $\mathbf{x}_1$ and $\mathbf{x}_2$ are feasible solutions. However, $\mathbf{x}_0 \frac{(\mathbf{x}_1+\mathbf{x}_2)}{2}$ which is impossible if $\mathbf{x}_0$ is an extreme point. Thus $r \leq m$. Also, $\mathbf{a}_1, \ldots \mathbf{a})_r$ are linearly independent, or else the above argument shows that $\mathbf{x}_0$ is not an extreme point. If $r = m$, then $\mathbf{x}_0$ is a basic feasible solution. If $r < m$, it is possible to select $m - r$ columns of A which, together with $\mathbf{a}_1, \ldots \mathbf{a}_r$ are linearly independent. The corresponding variables define a basic feasible solution.

(b) If $max\ z' < 0$, then the problem is infeasible, and phase 2 is not executed. If $max\ z' = 0$, then phase 2 is performed but with modifications to ensure that artificial basic variables keep their current zero values. For example, if in the pivot column some row has an artificial basic variable which has a negative entry, then this is chosen as the pivot element. Othervise the normal ratios determine the pivot row.

(c) An equivilent problem results if we maximize

$$z' = -16x_1 - 15x_2 - 4x_3$$

Introduce slack variables $s_1 \geq 0,\ s_2 \geq 0$

Basic	z'	x_1	x_2	x_3	s_1	s_2	
s_1	0	-2	-4	-5	1	0	-25
s_2	0	-8	-3	4	0	1	-28
	1	16	15	4	0	0	0
Ratio		2	5				

Basic	z'	x_1	x_2	x_3	s_1	s_2	
s_1	0	0	$-\frac{13}{4}$	-6	1	$-\frac{1}{4}$	-18
x_1	0	1	$\frac{3}{8}$	$-\frac{1}{2}$	0	$-\frac{1}{8}$	$\frac{7}{2}$
	1	0	9	12	0	2	-56
Ratio			$\frac{36}{13}$	$\frac{4}{3}$		8	

Basic	z'	x_1	x_2	x_3	s_1	s_2	
x_3	0	0	$\frac{13}{24}$	1	$-\frac{1}{6}$	$\frac{1}{24}$	3
x_1	0	1	$\frac{31}{48}$	0	$-\frac{1}{12}$	$-\frac{5}{48}$	5
	1	0	$\frac{5}{2}$	0	2	$\frac{3}{2}$	-92

Optimal solution $x_1 = 5,\ x_2 = 0,\ x_3 = 3,\ z = 92$