Question

Describe briefly what is meant by a linear birth-death process.

Amoeba,a single cell animal, reproduces itself by dividing into two. A flask of water contains a number, b, of amoeba. The probability that an amoeba divides into two in a time interval of length δt is $\lambda\delta t + o(\delta t)$, and the probability that it dies is $\mu\delta t + o(\delta t)$. Let $p_n(t)(n = 0,\ 1,\ 2,\ \cdots)$ denote the probability that the flask contains n amoebae at times t, and $p'_n(t)$ denote its derivative with respect to time. Show that

$p'_n(t) = \lambda(n-1)p_{n-1}(t) - (\lambda+\mu)np_n(t) + \mu(n+1)p_{n+1}(t)$,

$n = 1,\ 2,\ 3,\ \cdots$.

Suppose that the mean number of amoebae at time t is

$$M(t) = \sum_{n=0}^{\infty} np_n(t).$$

Show that $M(t)$ satisfies the differential equation

$$M'(t) = (\lambda - \mu)M(t),$$

and hence find $M(t)$.

If $W(t)$ denotes the mean of the square of the number of amoebae at time t prove that

$$W'(t) = 2(\lambda-\mu)W(t) + (\lambda+\mu)M(t).$$

Explain, without performing any calculations, how the result could be used to find the variance of the number of amoebae at time t.

Answer

A linear birth-death process is a $s.p(X(t) : t \geq 0)$ where $X(t)$ is the number of individuals in the population at time t, and where, in any time interval of length δt each individual has, independent of age and other individuals, a probability $\lambda\delta t + o(\delta t)$ of producing a new individual, and a probability $\mu\delta t + o(\delta t)$ of dying

$P(X(t+\delta t) = n+1 \mid X(t) = n) = \lambda n\delta t + o(\delta t)$

$P(X(t+\delta t) = n-1 \mid X(t) = n) = \mu n\delta t + o(\delta t)$ as δt → 0

$P(X(t+\delta t) = n \mid X(t) = n) = 1 - (\lambda+\mu)n\delta t + o(\delta t)$

$$\begin{aligned} P_n(t+\delta t) &= P(X(t+\delta t) = n) \\ &= P(X(t+\delta t) = n \mid X(t) = n-1)P(X(t) = n-1) \end{aligned}$$

$$
\begin{aligned}
& +P(X(t+\delta t) = n \mid X(t) = n+1)P(X(t) = n+1) \\
& +P(X(t+\delta t) = n \mid X(t) = n)P(X(t) = n) \\
= \;& \lambda(n-1)\delta t p_{n-1}(t) + \mu(n+1)\delta t p_{n+1}(t) \\
& +(1-(\lambda+\mu)n\delta t)p_n(t) + o(\delta t)
\end{aligned}
$$

Thus

$\dfrac{p_n(t+\delta t) - p_n(t)}{\delta t} =$

$\lambda(n-1)p_{n-1}(t) + \mu(n+1)p_{n+1}(t) + \mu(n+1)p_{n+1}(t) - (\lambda+\mu)np_n(t)$

Now $M(t) = \sum_{n=0}^{\infty} np_n(t) = \sum_{n=1}^{\infty} np_n(t)$

so $M'(t) = \sum_{n=1}^{\infty} np'_n(t)$

$= \sum_{n=1}^{\infty} \lambda(n-1)np_{n-1}(t) + \sum_{n=1}^{\infty} \mu(n+1)np_{n+1}(t) - \sum_{n=1}^{\infty}(\lambda+\mu)n^2p_n(t)$

$= \sum_{n=0}^{\infty} \lambda n(n-1)np_n(t) + \sum_{n=0}^{\infty} \mu n(n-1)np_n(t) - \sum_{n=0}^{\infty}(\lambda+\mu)n^2p_n(t)$

$= \sum_{n=0}^{\infty} p_n(t)[\lambda n^2 + \lambda n + \mu n^2 - \mu n - \lambda n^2 - \mu n^2]$

$= (\lambda-\mu)\sum_{n=0}^{\infty} np_n(t) = (\lambda-\mu)M(t)$

so $M'(t) = (\lambda-\mu)M(t)$

The general solution is $M(t) = Ae^{(\lambda-\mu)t}$

$X(0) = b$ so $M(0) = b$

Thus $M(t) = be^{(\lambda-\mu)t}$

Now $W(t) = \sum_{n=1}^{\infty} n^2p_n(t)$ so

$$
\begin{aligned}
W'(t) &= \sum_{n=1}^{\infty} n^2 p_n'(t) \\
&= \sum_{n=1}^{\infty} \lambda n^2 (n-1) p_{n-1}(t) + \sum_{n=1}^{\infty} \mu n^2 (n+1) p_{n+1}(t) \\
&\quad - \sum_{n=1}^{\infty} (\lambda+\mu) n^3 p_n(t) \\
&= \sum_{n=0}^{\infty} \lambda (n+1)^2 n(n-1) p_n(t) + \sum_{n=0}^{\infty} \mu (n-1)^2 n p_n(t) \\
&\quad - \sum_{n=0}^{\infty} (\lambda+\mu) n^3 p_n(t) \\
&= \sum_{n=0}^{\infty} p_n(t)[\lambda n^3 + 2\lambda n^2 + \lambda n + \mu n^3 \\
&\quad -2\lambda\mu^2 + \mu n - \lambda n^3 - \mu n^3] \\
&= 2(\lambda-\mu) \sum_{n=0}^{\infty} n^2 p_n(t) + (\lambda+\mu) \sum_{n=0}^{\infty} np-n(t)
\end{aligned}
$$

Thus $W'(t) = 2(\lambda-\mu)W(t) + (\lambda+\mu)M(t)$

Since $M(t)$ is known, this is a linear 1st order equation which can be solved for $W(t)$.

Then $Var(t) = W(t) - M9t)^2$.