**Question**

**(a)** A gambler with initial capital $z$ plays a sequence of games against an opponent with initial capital $(a-z)$. At each game the gambler wins 1 from his opponent with probability $p$, loses 1 to him with probability $q$ or the game is drawn and neither player gains any money. The games are independent with $p+q<1$ and $0<p<q<1$.

Obtain a difference equation for the probability $p_z$, that the gambler will ruin his opponent eventually, together with two boundary conditions. Solve this equation and use it to find the effect of trebling the initial capitals on the gambler's chance of ruining his opponent.

**(b)** For a Poisson process $(N(t); t \geq 0)$ with rate $\lambda$, write down the probability that $N(t)$ equals $n$ $(n = 0,\ 1,\ 2, \cdots\ ;\ t > 0)$.

Prove that for any $s$ and $t$ such that $0leqs < t,\ N(t) - N(s)$ has a Poisson distribution with parameter $\lambda(t-s)$.

**Answer**

**(a)** $p_z = p \cdot p_{z+1} + qp_{z-1} + (1-p-q)p_z$

so $p_z = \dfrac{p}{p+q}p_{z+1} + \dfrac{q}{p+q}p_{z-1}$

with boundary conditions $p_0 = 0,\ p_a = 1$

Substituting $p_z = \lambda^z$ gives

$p_z = \dfrac{p}{p+q}\lambda^{z+1} + \dfrac{q}{p+q}\lambda^{z-1}$

i.e. $\dfrac{p}{p+q}\lambda^2 - \lambda - \dfrac{q}{p+q} = 0$

$(\lambda - 1)\left(\dfrac{p}{p+q}\lambda - \dfrac{q}{p+q}\right) = 0$

so $\lambda = 1$ or $\lambda = \dfrac{q}{p}$

The general solution is

$$p_z = A\left(\frac{q}{p}\right)^z + B$$

The boundary conditions give

$0 = A + B \quad 1 = A\left(\frac{q}{p}\right)^a + B$

so $A = \dfrac{1}{(\frac{q}{p})^a - 1}$ and $B = -A$

so $p_z = \dfrac{(\frac{q}{p})^z - 1}{(\frac{q}{p})^a - 1}$

Replacing $z$ and $a$ by $3z$ and $3a$ gives

$p_{3z} = \dfrac{(\frac{q}{p})^{3z} - 1}{(\frac{q}{p})^{3a} - 1} = p_z \times \dfrac{(\frac{q}{p})^{2z} + (\frac{q}{p})^z + 1}{(\frac{p}{q})^{2a} + (\frac{q}{p})^a + 1} < p_z$ for $z < a$ and $q > p$.

so the chances of the opponent being ruined are reduced.

**(b)** $P(N(t) = n) = \dfrac{(\lambda t)^n}{n!} e^{-\lambda t} \quad n = 0\,1,\ 2,\ \cdots$

$$\begin{aligned}
P(N(t) - N(s) = n) &= \sum_{m=0}^{\infty} P(N(s) = m \\
&\quad \text{and N(t) = m + n)} \\
&= \sum_{m=0}^{\infty} P(N(s) = m \\
&\quad \text{and N(t − s) = n)} \\
&= \sum_{m=0}^{\infty} P(N(s) = m) P(N(t-s) = n) \\
&\quad \text{(Numbers of events in non-} \\
&\quad \text{overlapping time intervals indep.)} \\
&= P(N(t-s) = n) \sum_{m=0}^{\infty} \underbrace{P(N(s) = m)}_{=1} \\
&= P(N(t-a) = n)
\end{aligned}$$

so $N(t-s)$ is Poisson with rate $\lambda(t-s)$.