

### Question

Define a stationary distribution of a Markov chain.

The following is a modification of the Ehrenfest model of the exchange of heat or of gas molecules between two isolated bodies.

Suppose there are two boxes, labelled  $A$  and  $B$ , and  $N$  balls labelled  $1, 2, 3, \dots, N$ .

Initially some of the three balls are in box  $A$  and the remainder are in box  $B$ . A trial is performed in which an integer is selected at random from  $1, 2, 3, \dots, N$  and the ball labelled by that integer is removed from its box; one of the boxes is selected at random and the removed ball is placed in this box. The trails are repeated indefinitely, and the selections made are independent. If  $X_n$  denotes the number of balls in box  $A$  after the  $n$ th trial, explain why  $\{X_n\}_{n=0,1,\dots}$  constitutes a Markov chain and write down its state space.

Show that the conditional probability of  $(i + 1)$  balls in box  $A$  after the  $n$ th trial, given there are  $i$  balls in  $A$  after the previous trial and  $0 \leq i \leq n$ , is

$$\frac{N - i}{2N}$$

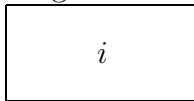
Obtain the remaining 1-step transition probabilities. Show that the stationary distribution is Binomial with parameters  $N$  and  $\frac{1}{2}$ .

Deduce the mean number of trials between occasions on which box  $A$  is empty.

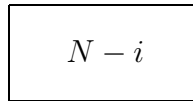
### Answer

If a Markov chain has transition matrix  $M$ , then the probability vector  $\underline{\pi}$  is a stationary distribution if  $\underline{\pi}M = \underline{\pi}$ . If  $X_n$  denotes the number of balls in Box  $A$  at step  $n$ , the number of balls in Box  $A$  at step  $n + 1$  depends only on the location of the balls at stage  $n$ , and the next choice of box. It does not depend on hoe stage  $n$  was arrived at. The state space is  $\{0, 1, 2, \dots, N\}$ .

Stage  $n$ :



Box  $A$



Box  $B$

after the next change there will be  $i + 1$  balls in box  $A$  if the integer chosen corresponds to a ball in box  $B$ , which happens with probability  $\frac{(N - i)}{N}$ , and if box  $A$  is selected, which happens with probability  $\frac{1}{2}$ . So

$$p_{i, i+1} = \frac{1}{2} \frac{N-i}{N} \quad 0 \leq i \leq N$$

Similarly

$$p_{i, i-1} = \frac{1}{2} \frac{i}{N} \quad 0 \leq i \leq N$$

$$p_{01} = 1 - p_{00} = 1 - \frac{1}{2} = \frac{1}{2} \quad \text{and} \quad p_{NN} = \frac{1}{2}$$

$$\text{For } 0 < i < N, \quad p_{ii} = 1 - p_{i, i+1} - p_{i, i-1} = \frac{1}{2}.$$

So the transition matrix is  $(N+1) \times (N+1)$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2} \frac{1}{N} & \frac{1}{2} & \frac{1}{2} \frac{N-1}{N} & 0 & \cdots & & \\ 0 & \frac{1}{2} \frac{2}{N} & \frac{1}{2} & \frac{1}{2} \frac{N-2}{N} & \cdots & & \\ \vdots & & & & & & \\ 0 & \cdots & & \frac{1}{2} \frac{N-1}{N} & \frac{1}{2} & \frac{1}{2} \frac{1}{N} \\ 0 & \cdots & & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

So if  $\underline{\pi} (\pi_0, \dots, \pi_N)$  is a stationary distribution, it satisfies  $\underline{\pi}M = \underline{\pi}$ , giving the following equations

$$\begin{aligned} \pi_0 \frac{1}{2} + \pi_1 \frac{1}{2} \frac{1}{n} &= \pi_0 \\ \pi_0 \frac{1}{2} + \pi_1 \frac{1}{2} + \pi_2 \frac{1}{2} \frac{2}{N} &= \pi_1 \\ \pi_1 \frac{1}{2} \frac{N-1}{N} + \pi_2 \frac{1}{2} + \pi_3 \frac{1}{2} \frac{3}{N} &= \pi_2 \\ \vdots \\ \pi_{k-1} \frac{1}{2} \frac{N-(k-1)}{N} + \pi_k \frac{1}{2} + \pi_{k+1} \frac{1}{2} \frac{k+1}{N} &= \pi_k \\ \vdots \\ \pi_{N-1} \frac{1}{2} \frac{1}{N} + \pi_N \frac{1}{2} &= \pi_N \end{aligned}$$

i.e.

$$\begin{aligned} \pi_0 &= \pi_1 \frac{1}{N} \\ \pi_1 &= \pi_0 + \pi_2 \frac{2}{N} \\ \vdots \\ \pi_N &= \pi_{N-1} \cdot \frac{1}{N} \end{aligned}$$

We need to prove by induction that

$$\pi_k = \binom{N}{k} \pi_0$$

True for  $k = 0, 1$ .

For  $1 < k < N$

$$\begin{aligned} \pi_{k+1} &= \frac{N}{k+1} \left( \pi_k - \pi_{k-1} \frac{N-k+1}{N} \right) \\ &= \frac{N}{k+1} \left( \binom{N}{k} - \binom{N}{k-1} \frac{N-k+1}{N} \right) \pi_0 \end{aligned}$$

which reduces to  $\binom{N}{k+1} \pi_0$ .

Finally  $\pi_N = \frac{1}{N} \pi_{N-1}$  fits with these results.

The Markov chain is finite irreducible, so the standard deviation is the eq.dist.

$$\mu_0 = \frac{1}{\pi_0} = 2^N$$