Question
Define a stationary distribution of a Markov chain.
The following is a modification of the Ehrenfest model of the exchange of heat or of gas molecules between two isolated bodies.
Suppose there are two boxes, labelled A and B, and N balls labelled $1,\ 2,\ 3,\ \cdots,\ N$. Initially some of the three balls are in box A and the remainder are in box B. A trial is performed in which an integer is selected at random from $1,\ 2,\ 3,\ \cdots,\ N$ and the ball labelled by that integer is removed from its box; one of the boxes is selected at random and the removed ball is placed in this box. The trails are repeated indefinitely, and the selections made are independent. If X_n denotes the number of balls in box A after the nth trial, explain why $\{X_n\}_{n=0,1,\cdots}$ constitutes a Markov chain and write down its state space.
Show that the conditional probability of $(i+1)$ balls in box A after the nth trial, given there are i balls in A after the previous trial and $0 \leq i \leq n$, is

$$\frac{N-i}{2N}$$

Obtain the remaining 1-step transition probabilities. Show that the stationary distribution is Binomial with parameters N and $\frac{1}{2}$.
Deduce the mean number of trials between occasions on which box A is empty.

Answer
If a Markov chain has transition matrix M, then the probability vector $\underline{\pi}$ is a stationary distribution if $\underline{\pi}M = \underline{\pi}$. If X_n denotes the number of balls in Box A at step n, the number of balls in Box A at step $n+1$ depends only on the location of the balls at stage n, and the next choice of box. It does not depend on hoe stage n was arrived at. The state space is $\{0,\ 1\,2,\ \cdots,\ N\}$.
Stage n:

i	$N-i$
Box A	Box B

after the next change there will be $i+1$ balls in box A if the integer chosen corresponds to a ball in box B, which happens with probability $\frac{(N-i)}{N}$, and if box A is selected, which happens with probability $\frac{1}{2}$. So

$$p_{i,\ i+1} = \frac{1}{2}\frac{N-i}{N} \quad 0 \leq i \leq N$$

Similarly

$$p_{i,\ i-1} = \frac{1}{2}\frac{i}{N} \quad 0 \leq i \leq N$$

$p_{01} = 1 - p_{01} = 1 - \dfrac{1}{2} = \dfrac{1}{2}$ and $p_{NN} = \dfrac{1}{2}$

For $0 < i < N,\ p_{ii} = 1 - p_{i,\ i+1} - p_{i,\ i-1} = \dfrac{1}{2}$.

So the transition matrix is $N+1 \times N+1)$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2}\frac{1}{N} & \frac{1}{2} & \frac{1}{2}\frac{N-1}{N} & 0 & \cdots & & \\ 0 & \frac{1}{2}\frac{2}{N} & \frac{1}{2} & \frac{1}{2}\frac{N-2}{N} & \cdots & & \\ \vdots & & & & & & \\ 0 & \cdots & & & \frac{1}{2}\frac{N-1}{N} & \frac{1}{2} & \frac{1}{2}\frac{1}{N} \\ 0 & \cdots & & & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

So if $\underline{\pi}$ $(\pi_0, \cdots,\ \pi_N)$ is a stationary distribution, it satisfies $\underline{\pi}M = \underline{\pi}$, giving the following equations

$\pi_0 \frac{1}{2} + \pi_1 \frac{1}{2}\frac{1}{n} = \pi_0$

$\pi_0 \frac{1}{2} + \pi_1 \frac{1}{2} + \pi_2 \frac{1}{2}\frac{2}{N} = \pi_1$

$\pi_1 \frac{1}{2}\frac{N-1}{N} + \pi_2 \frac{1}{2} + \pi_3 \frac{1}{2}\frac{3}{N} = \pi_2$

$\vdots$

$\pi_{k-1} \frac{1}{2}\frac{N-(k-1)}{N} + \pi_k \frac{1}{2} + \pi_{k+1} \frac{1}{2}\frac{k+1}{N} = \pi_k$

$\vdots$

$\pi_{N-1} \frac{1}{2}\frac{1}{N} + \pi_N \frac{1}{2} = \pi_N$

i.e.

$$\begin{aligned} \pi_0 &= \pi_1 \frac{1}{N} \\ \pi_1 &= \pi_0 + \pi_2 \frac{2}{N} \\ &\vdots \\ \pi_N &= \pi_{N-1} \cdot \frac{1}{N} \end{aligned}$$

We need to prove by induction that

$$\pi_k = \binom{N}{k} \pi_0$$

True for $k = 0,\ 1$.

For $1 < k < N$

$$\begin{aligned} \pi_{k+1} &= \frac{N}{k+1}\left(\pi_k - \pi_{k-1}\frac{N-k+1}{N}\right) \\ &= \frac{N}{k+1}\left(\binom{N}{k} - \binom{N}{k-1}\frac{N-k+1}{N}\right)\pi_0 \end{aligned}$$

which reduces to $\binom{N}{k+1}\pi_0$.

Finally $\pi_N = \frac{1}{N}\pi_{N-1}$ fits with these results.

The Markov chain is finite irreducible, so the standard deviation is the eq.dist.

$\mu_0 = \frac{1}{\pi_0} = 2^N$