QUESTION

Show that the functions

i) $y^3 - 3x^2y$ and

ii) $\frac{y}{(x^2+y^2)}$

are harmonic in some region of the plane. In each case find a conjugate harmonic function and identify the corresponding analytic function.

ANSWER

The Cauchy inequalities imply that if z_0 is a complex number, and if M_R is the maximum value of $|f(z)|$ on the circle centre z_0, radius R, then $|f^{''}(z_0)| \leq 2M_R/R^2$. As f is entire, (analytic thoughout $\mathbf{C}$), R can be as large as we please, so that $f^{''}(w) = 0$, for all $w \in \mathbf{C}$. Thus $f^{'}(w) = a$, a complex constant. Therefore

$$\int_0^z f^{'}(w)dw = az + b$$

(b a complex constant) and hence

$$f(z) = az + k$$

(k a complex constant). As $|f(z)| \leq A|z|$, by putting $z = 0$, we get $f(0) = 0$, so that $k = 0$ and $f(z) = az$.