QUESTION

If $f(z) = u(x,y) + iv(x,y)$ and $\overline{f(z)} = u(x,y) - iv(x,y)$ are both analytic in a region D show that f is constant in D. (Hint: Cauchy-Riemann equations). Also show that if f is analytic and $|f|$ is constant then f is constant.

ANSWER

(i) Either do this directly using the Cauchy-Riemann equations, or note that $(x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$. Thus $y^3 - 3xy$ is the imaginary part of $-z^3$ and so $y^3 - 3x^2y$ is harmonic, its conjugate harmonic function is $3xy^2 - x^3$ and the corresponding analytic function is $-z^3$. (ii) $1/z = 1/(x+iy) = (x-iy)/(x^2+y^2)$. Thus $y/(x^2+y^2)$ is the imaginary part of $-1/z$ and is thus harmonic in any region not containing the origin. Thus $y/(x^2+y^2)$ is the imaginary part of $-1/z$, so that $y/(x^2+y^2)$ is harmonic, its conjugate harmonic function is $-x/(x^2+y^2)$ and $-1/z$ is the corresponding analytic function.