Question

A population consists initially of b individuals. At any subsequent time an individual in the population has, during any time interval of length δt, independent of previous history and of other individuals,

(i) a probability $\alpha\delta t + o(\delta t)$ of producing a single offspring,

(ii) a probability $\beta\delta t + o(\delta t)$ of producing twins,

(iii) a probability $o(\delta t)$ of producing more than two offspring, as $\delta t \to 0$.

Let $p_n(t)$ denote the probability that the total population size is n at time t. Show that for $n = 1,\ 2,\ \cdots$,

$$p_n'(t) = -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) + \beta(n-2)p_{n-2}(t).$$

Find the probability that, at time t, the population has not changed from its original size.
Suppose that the mean number of individuals at time t is

$$M(t) = \sum_{n=0}^{\infty} np_n(t).$$

Show that $M'(t) = (\alpha + 2\beta)M(t)$ and hence find $M(t)$.

Answer

$P(X(t+\delta t) = n+1 \mid X(t) = n) = \alpha n\delta t + o(\delta t)$
$P(X(t+\delta t) = n+2 \mid X(t) = n) = \beta n\delta t + o(\delta t)$
$P(X(t+\delta t) = n \mid X(t) = n) = 1 - (\alpha+\beta)n\delta t + o(\delta t)$

Now
$p_b(t+\delta t) = p_b(t)(1-(\alpha+\beta)b\delta t + o(\delta t))$
$p_{b+1}(t+\delta t) = p_{b+1}(t)(1-(\alpha+\beta)(b+1)\delta t + o(\delta t) + p_b(t)(\alpha b\delta t + o(\delta t))$
For $n > b+1$

$$\begin{aligned} p_n(t+\delta t) &= p_n(t)(1-(\alpha+\beta)n\delta t + o(\delta t)) \\ &\quad + p_{n-1}(t)(\alpha(n-1)\delta t + o(\delta t)) \\ &\quad p_{n-2}(t)(\beta(n-2)\delta t + o(\delta t)) \end{aligned}$$

In fact, since $p_n(t) = 0$ for $n < b$, this equation encompasses the first two.
Thus for $n = 1,\ 2,\ \cdots$

$$\begin{aligned} \frac{p_n(t+\delta t) - p_n(t)}{\delta t} &= -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) \\ &\quad + p(n-2)p_{n-2}(t) + o(\delta t) \end{aligned}$$

Letting $\delta t \to 0$ gives
$p_n'(t0 = -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) + \beta(n-2)p_{n-2}(t)$

Now $P_b(0) = 1$ and $p_n(0) = 0$ for $n \neq b$.

$P_b'(t) = -(\alpha+\beta)bp_b(t)$
so $p_b(t) = Ke^{-(\alpha+\beta)bt}$
and using $p_b(0) = 1$ gives $K = 1$.

Hence $p_b(t) = e^{-(\alpha+\beta)bt}$.

$n = b+1$

$$\begin{aligned} p_{b+1}(t+\delta t) &= p_{b+1}(t)(1-(\alpha+\beta)(b+1)\delta t) \\ &\quad +p_b(t)(\alpha b\delta t) \\ \\ p_{b+1}'(t) &= -(\alpha+\beta)(b+1)p_{b+1}(t) + \alpha b p_b(t) \end{aligned}$$

$n > b+1$

$$\begin{aligned} p_n(t+\delta t) &= p_n(t)(1-(\alpha+\beta)n\delta t) \\ &\quad +p_{n-1}(t)(\alpha(n-1)\delta t) + p_{n-2}(t)(\beta(n-2)\delta t) \\ \\ p_n'(t) &= -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) \\ &\quad +\beta(n-2)p_{n-2}(t) \end{aligned}$$

Now $M(t) = \sum_{n=0}^{\infty} np_n(t)$

$$\begin{aligned} M'(t) &= \sum_{n=0}^{\infty} np_n'(t) \\ &= \sum_{n=0}^{\infty} n[-(\alpha+\beta)np_n(t) + \alpha(m-1)p_{n-1}(t) \\ &\quad +\beta(n-2)p_{n-2}(t)] \\ &= \sum_{n=0}^{\infty} -n^2(\alpha+\beta)p_n(t) + \sum_{n=0}^{\infty} \alpha n(n-1)p_{n-1}(t) \\ &\quad + \sum_{n=0}^{\infty} \beta n(n-2)p_{n-2}(t) \\ &= \sum_{n=0}^{\infty} -n^2(\alpha+\beta)p_n(t) + \sum_{n=0}^{\infty} \alpha n(n+1)np_n(t) \\ &\quad + \sum_{n=0}^{\infty} \beta(n+2)np_n(t) \\ &= \sum_{n=0}^{\infty} np_n(t)[-n(\alpha+\beta) + \alpha(n+1) + \beta(n+2)] \\ &= (\alpha+2\beta)M(t). \end{aligned}$$

Hence $M(t) = He^{(\alpha+2\beta)t}$

Now $M(0) = b$, and so

$$M(t) = be^{(\alpha+2\beta)t}$$