Question

Describe what is meant by a compound Poisson process. Show that if $A(z)$ is the probability generating function for the number of events occurring at each point of the process, then the random variable $X(t)$ - the total number of events occurring in a time interval of length t - has probability generating function

$$G_t(z) = \exp(\lambda t A(z) - \lambda t)$$

Points in time occur in a Poisson process with rate λ. At each point two fair coins are tossed. Find the probability generating function for the total number of heads occurring in a time interval of length t. Find the mean number of heads occurring in a time interval of length t.

Let W denote the waiting time before any heads occur.

Show that

$$P(W > t) = \exp\left(\frac{-3\lambda t}{4}\right)$$

Answer

Suppose that

(i) points occur in a Poisson process $\{N(t) : t \geq 0$ with rate λ

(ii) at the ith point Y_i events occur, where $Y_1,\ Y_2, \cdots$ are i.i.d random variables.

(iii) Y_i and $\{N(t) : t \geq 0\}$ are independent.

The total number of events occurring in a time interval of length t is

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

$\{X(t) : t \geq 0\}$ is said to be a compound Poisson process.

Let the p.g.f of each Y_i be $A(z)$. Then $X(t)$ has p.g.f

$$\begin{aligned}\sum_{j=0}^{\infty} z^j P(X(t) = j) &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} x^j P(X(t) = j \mid N(t) = n) P(N(t) = n) \\ & \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} z^j P(Y_1 + \cdots + Y_n = j) \frac{(\lambda t)^n e^{-\lambda t}}{n!}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty}\left\{\sum_{n=0}^{\infty} z^j P(Y_1+\cdots+Y_n=j)\right\}\frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty}[A(z)]^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&\quad \text{since the } Y_i \text{ are independent} \\
&= \exp(\lambda t A(z) - \lambda t)
\end{aligned}
$$

Now $A(x) = \frac{1}{4} + \frac{1}{2}x + \frac{1}{4}z^2$ in this case.
So the number of heads in a time interval of length t has p.g.f.

$$
\begin{aligned}
G_t(z) &= \exp\left(\lambda t\left(\frac{1}{4}+\frac{1}{2}z+\frac{1}{4}z^2\right)-\lambda t\right) \\
&= \exp\left(\lambda t\left(\frac{1}{4}z^2+\frac{1}{2}z-\frac{3}{4}\right)\right)
\end{aligned}
$$

The mean number of heads is $G'(1)$.

$$
G'(z) = \exp\left(\lambda t\left(\frac{1}{4}z^2+\frac{1}{2}z-\frac{3}{4}\right)\right)\cdot \lambda t\left(\frac{1}{2}z+\frac{1}{2}\right)
$$

so $G'(1) = e^0\lambda t = \lambda t$
Let W be the waiting time before a head is recorded.

<u>EITHER</u>

$$
\begin{aligned}
P(W>t) &= \text{P(no events in compound process in } (0,\ t]) \\
&= G_t(0) = \exp\left(-\frac{3}{4}\lambda t\right)
\end{aligned}
$$

<u>OR</u>

$$
\begin{aligned}
P(W>t) &= \text{P(no events in Poisson process)} \\
&\quad + \sum_{n=1}^{\infty} (n \text{ events and 2 tails at each event}) \\
&= e^{-\lambda t} + \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= e^{-\lambda t} + e^{-\lambda t}\left[e^{\lambda\frac{t}{4}} - 1\right] \\
&= e^{-\frac{3\lambda t}{4}}
\end{aligned}
$$