Question

Explain what a branching Markov chain is, and state clearly the Fundamental Theorem for branching chains.

Amoeba reproduce by dividing into two. Some amoeba will die without dividing. Assume that amoeba reproduce identically and independently and that an amoeba dies with probability p. Show that if $p \geq \frac{1}{2}$ then eventual extinction is effectively certain, irrespective of the initial population size. Calculate the probability of eventual extinction for $p < \frac{1}{2}$, for a population of initial size n.

A related organism can reproduce by multiple division, giving k offspring with probability a_k. The mean number of offspring per organism is $m > 1$. Explain how you would calculate the probability P of eventual extinction if the equation of which P is a root cannot be solved algebraically. Explain in outline how your approach can be justified.

Answer

Suppose we have a population of individuals, each reproducing independently of the others. Suppose the distributions of the number of offspring of all individuals are identical.

Let X_n be the number of individuals in the nth generation; then (X_n) is a branching Markov chain.

The Fundamental Theorem.

(i) The probability of extinction when the population has size 1 initially is the smallest positive root of the equation $x = A(x)$, where A is the p.g.f of the number of offspring per individual.

(ii) Extinction occurs with probability 1 if and only if $\mu \leq 1$, where μ is the mean number of offspring per individual.

An amoeba produces no offspring with probability p and 2 offspring with probability $(1-p)$.

The p.g.f is $A(s) = p + (1-p)s^2$

$\mu = 2(1-p) \leq 1$ for $p \geq \frac{1}{2}$

$x = A(x)$ gives
$$\begin{aligned} (1-p)x^2 - 2 + p &= 0 \\ (x-1)((1-p)x - p) &= 0 \end{aligned}$$

so $x = \frac{p}{1-p}$ is the smaller root for $p < \frac{1}{2}$.

By independence, if the population has initial size n, the probability of extinction is

$$\left(\frac{p}{1-p}\right)^n$$

Now let the p.g.f be $A(x)$. To find the probability of extinction we have to solve $x = A(x)$. In proving the fundamental theorem it is shown that $A(x)$ is convex downwards, and that it has at most two roots in $[0, 1]$, one of them being $x = 1$.
The other root in this case is positive, for $x = 0$ is not a root as is easily seen. Since $\mu > 1$, the root is < 1, and so the graph of $A(x)$ will be as follows
PICTURE

An iterative approach, with $0 < x_0 < 1$, using $x_n = A(x_{n-1})$ will therefore converge to P. In general it will be a 1st order process.