**Question**

The telephone directory for Hirstville consists of three volumes, $I$, $II$, $III$. The local Albanian take-away keeps is set of volumes in a neat pile by the telephone. When a volume us consulted it is always replaced on the top of the pile. The proprietor observes that over a long period of time volume $I$ is consulted with probability $p$, volume $II$ with probability $q$ and volume $III$ with probability $r$, where

$$p + q + r - 1,\ 0 < p < 1,\ 0 < q < 1,\ 0 < r < 1.$$

There are six possible orders for the pile of three volumes. Explain briefly why these form the six states of a Markov chain.

Draw a transition diagram and use it to explain why the Markov chain is ergodic. Write down the transition matrix. Find the equilibrium distribution. What are the mean recurrence times for each state?

**Answer**

The six states are:

| | | | |
|---|---|---|---|
| 1 | $I$ | $II$ | $III$ |
| 2 | $III$ | $I$ | $II$ |
| 3 | $II$ | $III$ | $I$ |
| 4 | $I$ | $III$ | $II$ |
| 5 | $II$ | $I$ | $III$ |
| 6 | $III$ | $II$ | $I$ |

The probability of transition between states does not depend on past history. In fact it depends only on which volume is chosen at each stage. This is the Markov property and so we have a Markov chain.

Transition diagram:

PICTURE

Since $p_{ii} > 0$ for each $i$ the chain is aperiodic. The circuit 1-2-4-5-6-3-1 show that all states intercommunicate. So the chain is irreducible and aperiodic i.e. ergodic, and so all states are positive recurrent.

Transition Matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} p & r & 0 & 0 & q & 0 \\ 0 & r & q & p & 0 & 0 \\ p & 0 & q & 0 & 0 & r \\ 0 & r & 0 & p & q & 0 \\ p & 0 & 0 & 0 & q & r \\ 0 & 0 & q & p & 0 & r \end{pmatrix} \end{array}$$

Equilibrium distribution. Because the Markov chain is ergodic there is a unique stationary distribution equal to the equilibrium distribution. The stationary distribution satisfies $\underline{\pi}P = \underline{\pi}$.
Thus we have

$$\begin{aligned} p(\pi_1 + \pi_3 + \pi_5) &= \pi_1 && (1) \\ p(\pi_1 + \pi_2 + \pi_4) &= \pi_2 && (2) \\ p(\pi_2 + \pi_3 + \pi_6) &= \pi_3 && (3) \\ p(\pi_2 + \pi_4 + \pi_6) &= \pi_4 && (4) \\ p(\pi_1 + \pi_4 + \pi_5) &= \pi_5 && (5) \\ p(\pi_3 + \pi_5 + \pi_6) &= \pi_6 && (6) \end{aligned}$$

$$\text{Adding 1 to 4, using } \sum \pi_i = 1, \text{ gives } \pi_1 + \pi_4 = p \qquad (7)$$

$$\text{Adding 3 to 5, using } \sum \pi_i = 1, \text{ gives } \pi_3 + \pi_5 = q \qquad (8)$$

$$\text{Adding 2 to 6, using } \sum \pi_i = 1, \text{ gives } \pi_2 + \pi_6 = r \qquad (9)$$

Substituting in 1 using 8 gives $p(\pi_1 + q) = \pi_1$ so $\pi_1 = \dfrac{pq}{1-p}$
Similarly we obtain
$\pi_2 = \dfrac{pr}{1-r}$, $\pi_3 = \dfrac{qr}{1-q}$, $\pi_4 = \dfrac{pr}{1-p}$ $\pi_5 = \dfrac{pq}{1-q}$, $\pi_6 = \dfrac{qr}{1-r}$
The mean recurrence times for ergodic Markov chains are the reciprocals of the equilibrium probabilities
so $\mu_1 = \dfrac{1-p}{pq}$ etc.