Question

A simple random walk has the infinite set $(a,\ a-1,\ a-2,\ \cdots)$ as possible states. State a is an upper reflecting barrier, for which reflection is certain, i.e., if the random walk is in state a at step n then it will be in state $a-1$ at step $n+1$. For all other states, transitions of $+1,\ -1,\ 0$ take place with probabilities $p,\ q,\ 1-p-q$ respectively.

Let $p_{j,\ k}^{(n)}$ denote the probability that the random walk is in state k at step n, having starter in state j. Obtain difference equations relating to these probabilities, for $k=a,\ k=a-1$ and $k<1-1$.

Assuming that there is a long-term equilibrium distribution $\pi_k)$, where

$$\pi_k = \lim_{n\to\infty} p_{j,\ k}^{(n)} \text{ for j = a, a}-1,\ \text{a}-2,\ \cdots\ ,$$

use the difference equations for $p_{j,\ k}^{(n)}$ to obtain a set of difference equations for π_k for $k<a-1$, and also deduce that

$$p\pi_{a-2} = q\pi_{a-1} \text{ and } \mathrm{p}\pi_{\mathrm{a}-1} = \pi_{\mathrm{a}}.$$

Solve this set of equations for the case $p<q$ to obtain explicit expressions for π_k in terms of $p,\ q$ and a.

Answer

$$\begin{array}{lrcl} k=a: & p_{j\ a}^{(n)} & = & p\,p_{j\ a-1}^{(n-1)} \\ k=a-1: & p_{j\ a-1}^{(n)} & = & p\,p_{j\ a-2}^{(n-1)} + p_{j\ a}^{(n-1)} + (1-p-q)p_{j\ a-1}^{(n-1)} \\ k<a-1: & p_{j\ k}^{(n)} & = & p\,p_{j\ k-1}^{(n-1)} + q\,p_{j\ k+1}^{(n-1)} + (1-p-q)p_{j\ k}^{(n-1)} \end{array}$$

Assuming an equilibrium distribution (π_k), taking limits in the above equations gives

$$\begin{array}{rcl} \pi_a & = & p\pi_{a-1} \\ \pi_{a-1} & = & p\pi_{a-2} + \pi_a + (1-p-q)\pi_{a-1} \\ \pi_k & = & p\pi_{k-1} + q\pi_{k+1} + (1-p-q)\pi_k \text{ for k} < \text{a}-1 \end{array}$$

substituting the first equation in the second gives

$$p\pi_{a-2} = q\pi_{a-1}.$$

The solution of the difference equation for $k<a-1$ is $\pi_k = A\left(\frac{p}{q}\right)^k + B$.

Now if $B \neq 0$, $\sum_{\pi_k}$ diverges, so $B=0$

$\pi_{a-1} = \left(\frac{p}{q}\right)\pi_{a-2} = A\left(\frac{p}{q}\right)^{a-1}$

$\pi_a = p\pi_{a-1} = p \cdots A\left(\frac{p}{q}\right)^{a-1}$

We require $\sum \pi_k = 1$

i.e. $A\left[p\left(\frac{p}{q}\right)^{a-1} + \sum_{k=-\infty}^{a-1}\left(\frac{p}{q}\right)^{k}\right] = 1$

$A\left[\frac{p\cdot\left(\frac{p}{q}\right)^{a-1} + \left(\frac{p}{q}\right)^{a-1}}{(1-\frac{q}{p})}\right] = 1$

$A\left(\frac{p}{q}\right)^{a-1}\left[p + \frac{1}{1-\frac{q}{p}}\right] = 1$

$A = \left(\frac{p-q}{p^2-pq+p}\right)\left(\frac{q}{p}\right)^{a-1}$

Hence $\pi_a = \frac{p-q}{p-q+1}$

and $\pi_k = \left(\frac{p-q}{p^2-pq+p}\right)\left(\frac{q}{p}\right)^{a-1-k}$ for $k < a$

Can also be done by recursion.