Question

A gambler with initial capital z plays against an opponent with capital $(z-1)$, where a and z are integers and $0 \leq z \leq a$. At each play the gambler wins 1 with probability p and loses 1 with probability $q = 1 - p$.

Let q_z denote the probability that the gambler will eventually be ruined. Write down a recurrent relation for q_z and solve it to obtain explicit formulae for q_z in terms of $z,\ a$, and p, in both cases $p = \dfrac{1}{2}$ and $p \neq \dfrac{1}{2}$.

Two players begin a game of dice with 10 each. At each play they both stake 1 and each of them throws a fair cubical die. If player A has a higher score than player B he wins, otherwise he loses. Player B says that if player A gets down to his last 5 he will give him a chance by changing the game to one of tossing a fair coin until one of the players us ruined. In this game A wins if the coin lands heads and B wins if it lands tails again with 1 stake. Calculate the probability that player A will eventually be ruined.

Answer

We argue conditionally on the result of the first play to obtain
$q_z = pq_{z+1} + qp_{z-1}$ where $q = 1 - p$
The auxiliary equation is

$$p\lambda^2 - \lambda + q = 0$$

i.e., $(p\lambda - q)(\lambda - 1)$ since $p + q = 1$.
so $\lambda = \dfrac{q}{p} \quad \lambda = 1$.
We have unequal roots of $q \neq p$.
Then $q_z = A + B\left(\dfrac{q}{p}\right)^z \quad$ for $0 < z < a$.
Boundary conditions are $q_0 = 1m\ q_a = 0$
and these give A and B

$$q_z = \frac{(\frac{q}{p})^a - (\frac{q}{p})^z}{(\frac{q}{p})^a - 1}$$

For $p = 1 = \dfrac{1}{2},\ q_z = Az + B$, and then boundary conditions give

$$q_z = 1 - \frac{z}{a}$$

For player A to be ruined, he must get down to 5 and then be ruined on the coin-tossing game.

To find the probability that he gets down to 5 is equivalent to playing the dice game where player A has 5 and player B has 10.

P(player A has a higher score than player B)=$\frac{1}{2}\left(\frac{36-6}{36}\right) = \frac{15}{36}$

so $p = \frac{15}{36},\ q = \frac{21}{6},\ z = 5,\ a = 15,\ \frac{q}{p} = \frac{21}{5} = \frac{7}{5}$

so P(player A gets down to 5)=$\frac{(\frac{7}{5})^{15} - (\frac{7}{5})^{5}}{(\frac{7}{5})^{15} - 1} = 0.97167\cdots$

The game now becomes a fair game with $z = 5$ and $a = 20$

so P(player A ruined in fair game)=$1 - \frac{5}{20} = \frac{3}{4}$

The plays throughout are independent so overall

P(A ruined)=$0.97167\cdots \times \frac{3}{4} = .728755771\cdots$