NOTE ANSWER IS NOT COMPLETE!!!!!

Question

Laplace's equation can be written in polar coordinates as

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Suppose $\nabla^2\phi = 0$ inside the unit semi-circle $0 < r \leq 1,\ 0 \leq \theta \leq \pi$. Deduce that a possible solution can take the form $\phi = A + B\theta$ where $A,\ B$ are constants. Confirm that this is indeed the unique solution when the following boundary conditions are specified and find A and B.

$$\frac{\partial\phi}{\partial r} = 0 \text{ when r} = 1;\quad \phi = +1 \text{ when } \theta = 0;\quad \phi = -1 \text{ when } \theta = \pi$$

Show that $\phi = Re(A - iB\log z),\ z = x + iy = r\exp(i\theta)$. Now use the method of conformal transformations to solve the following boundary-value problems, where $\nabla^2\phi = 0$ in the unit semi-circle $0 < r \leq 1,\ 0 \leq \theta \leq \pi$.

(a) If the boundary conditions are now:

$$\phi = -1 \text{ when r} = 1, 0 \leq \theta < \frac{\pi}{2};\quad \phi = +1 \text{ when r} = 1,\ \frac{\pi}{2} \leq \theta < \pi$$

$$\frac{\partial\phi}{\partial y} = 0 \text{ when } \theta = 0 \text{ or } \theta = \pi,\ 0 \leq \text{r} \leq 1$$

In particular, show that on $\theta = \pi,\ \phi = 1 - \frac{2}{\pi}\arctan\left(\frac{1-r^2}{2r}\right)$.

(Hint: consider the transformation $w = \frac{(z-i)}{(iz-1)}$.)

(b) If the boundary conditions are now

$\frac{\partial\phi}{\partial r} = 0$ when r $= 1,\ 0 \leq \theta < \pi$;

$\phi = -1$ when $\theta = \pi, 0 \leq$ r ≤ 1;

$\phi = -1$ when $\theta = 0,\ 0 \leq$ r $< \frac{1}{2}$;

$\phi = +1$ when $\theta = 0,\ \frac{1}{2} \leq$ r ≤ 1

In particular, show that on $r = 1$, for a suitably chosen range of arctan,

$$\phi = 1 - \frac{2}{\pi} \arctan\left(\frac{2\sin\theta}{5\cos\theta - 4}\right)$$

(Hint: consider the transformation $w = \dfrac{(2z-1)}{(2-z)}$.)

Answer
PICTURE

Check: $\phi(r,\theta) = A + B\theta \Rightarrow \phi_r = \phi_{rr} = 0,\ \phi_\theta = B,\ \phi_{\theta\theta} = 0$

Therefore $\nabla^2\phi(x,y) = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0 + \frac{0}{r} + \frac{0}{r^2} = 0,\ r \neq 0 \surd\surd$

Now satisfy boundary conditions

$\phi_r = 0$ at $r = 1 \Rightarrow \frac{\partial_r}{\partial r}(a + B\theta) = 0$

(which it does everywhere so boundary condition is satisfied)

$$\begin{aligned} \phi_r = +1 \text{ when } \theta = 0 &\Rightarrow B \times 0 + A \\ &\Rightarrow A = 1 \\ \phi = -1 \text{ when } \theta = \pi &\Rightarrow B\pi + A = -1 \\ &\Rightarrow B = -\frac{2}{\pi} \end{aligned}$$

so $\phi(r,\theta) = 1 - \frac{2}{\pi}\theta$ satisfies $\nabla^2\phi = 0$ in P

Consider $z = x + iy$

$\Phi(z) = A - iB\log z, \quad A,\ B \in \mathbf{R}$

$Re[\Phi(z)] = Re(A + iB\log|z| + B\theta) = A + B\theta = \phi(r,\theta)$ as required.

(a) PICTURE

consider $w = \left(\frac{z-i}{iz-1}\right)$:
$y = 0$ $0 \leq x \leq 1$:

$$\begin{aligned} w &= \frac{x-i}{ix-1} \\ &= \frac{(x-i)}{x^2+1}(-1-ix) \\ &= \frac{-x-ix^2+i-x}{x^2+1} \\ &= \frac{-2x-i(x^2-1)}{x^2+1} \end{aligned}$$

$$\begin{aligned} (iz-1) &= \left(\frac{z-i}{w}\right) \\ \left(i-\frac{1}{w}\right)^2 &= 1-\frac{i}{w} \\ z &= \frac{w-i}{(iw-1)} \end{aligned}$$

$$\begin{aligned} |iw-1| &= |w-i| \\ |w+i| &= |w-i| \end{aligned}$$

$z = Re^{i\theta}$
$w = \left(\frac{Re^{i\theta}-i}{iRe^{i\theta}-1}\right) : w = \left(\frac{R-i}{iR-1}\right) = \frac{R-i}{R+i}\frac{1}{i}$ where $\theta = 0$
$\frac{1}{i}\frac{(z-i)}{(z+i)} = \frac{1}{i}\frac{(z-i)^2}{|z|^2+1} = \frac{1}{i}\frac{1}{i}\frac{z^2-2iz-1}{(|z|^2+1)}$
$\frac{(Re^{i\theta}-i)}{i(Re^{i\theta}+i)} = \frac{1}{i}\frac{(Re^{2i\theta}-2iRe^{i\theta}-1)}{(R^2+1)}$

$$\begin{aligned} w(0) &= +i \\ w(i) &= 0 \\ w(1) &= \frac{1-i}{-(1+i)} = -1 \\ w(-1) &= \frac{-(1+i)}{-(1+i)} = +1 \end{aligned}$$

$$
\begin{aligned}
w &= \frac{(x-i)}{-1+ix} \\
&= \frac{(x-i)(-1-ix)}{1+x^2} \\
&= -x-ix^2+i-x \\
&= \frac{-2x+i(1-x^2)}{1+x^2}
\end{aligned}
$$