NOTE REFERENCE TO SHEET 5, QUESTION 8

Question

Reconsider the conformal map of Q8, exercises 5 and the example in the lecture notes. Find a solution ϕ of Laplace's equation $\nabla^2\phi = 0$ inside the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = \frac{1}{4}$, if $\phi = 0$ on th ellipse boundary and $\phi = V$ when $y = 0,\ |x| < 2$. This situation models the electric potential inside an ellipsoidal capacitor.

Answer

From Q8, Ex 5 we have:

PICTURE

Now solution of $\nabla^2\phi(u,v) = 0$ in concentric circles (see lecture notes) is

$$\begin{aligned}\phi_1(u,v) &= A\log|w| + B\\ &= A\log\sqrt{u^2+v^2} + b\\ &= Re[A\log w + B], \quad A, B, \in \mathbf{R}\end{aligned}$$

Boundary conditions on $|w| = r$,

On $r = 1,\ V = A\log 1 + B \Rightarrow B = V$

On $r = 2,\ 0 = A\log 2 + B \Rightarrow A = -\frac{V}{\log 2}$

Therefore

$$\begin{aligned}\phi_1(u,v) &= \left[V - \frac{V}{\log 2}\log|w|\right]\\ &= \frac{V}{\log 2}\log\left(\frac{2}{|w|}\right)\end{aligned}$$

Therefore $\Phi_1(w) = \frac{V}{\log 2}\log\left(\frac{2}{w}\right)$

So $\Phi(z) = \Phi_1(w) = \Phi_1\left(z + \frac{1}{z}\right)$
Hence back in the z-plane we have,

$$\Phi(z) = \frac{V}{\log 2} \log\left(\frac{2}{z + \frac{1}{z}}\right)$$

and so the harmonic solution $\nabla^2\phi(x,y) = 0$ which satisfies the boundary conditions is:

$$\begin{aligned}
\phi(x,y) &= Re[\Phi(z)] \\
&= \frac{V}{\log 2} \log\left|\frac{2}{z + \frac{1}{z}}\right| \\
&= \frac{V}{\log 2} \log\left|\frac{2z}{z^2 + 1}\right| \\
&= -\frac{V}{\log 2} \log\left|\frac{1}{2}\left(z + \frac{1}{z}\right)\right|
\end{aligned}$$

So setting $z = x + iy$, we get

$$\begin{aligned}
\left|\frac{1}{2}\left(z + \frac{1}{z}\right)\right| &= \frac{1}{2}\left|x + iy + \frac{x - iy}{x^2 + y^2}\right| \\
&= \frac{1}{2}\left|x\frac{(x^2 + y^2 + 1)}{x^2 + y^2} + iy\frac{(x^2 + y^2 - 1)}{x^2 + y^2}\right| \\
&= \frac{1}{2}\sqrt{x^2\frac{(x^2 + y^2 + 1)^2}{(x^2 + y^2)^2} + y^2\frac{(x^2 + y^2 - 1)^2}{(x^2 + y^2)^2}} \\
&= \frac{1}{2(x^2 + y^2)} \times \sqrt{x^2(x^2 + y^2 + 1)^2 + y^2(x^2 + y^2 - 1)^2}
\end{aligned}$$

so
$\phi(x,y)$
$= -\frac{V}{\log 2} \log\left[\frac{1}{2(x^2 + y^2)} \times \sqrt{x^2\frac{(x^2 + y^2 + 1)^2}{(x^2 + y^2)^2} + y^2\frac{(x^2 + y^2 - 1)^2}{(x^2 + y^2)^2}}\right]$

UGH!!