Question

Show that the transformation

$$w = \frac{i-z}{i+z}$$

maps the upper half plane $Im(z) > 0$ into the interior of the unit circle, centred on the origin of the w-plane. Hence, solve Laplace's equation $\nabla^2 F = 0$ inside the circle $w = exp(i\theta)$, when the values of F on the circumference are

$$F = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$$

Answer

$w = \dfrac{i-z}{i+z}$

Consider the <u>inverse</u> mapping $w \to z$.

If $|w| = 1$, i.e., the unit circle,

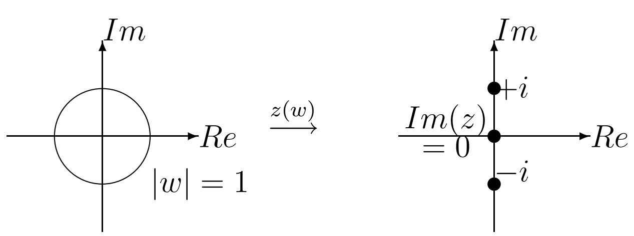
then

$$|w| = \left|\frac{i-z}{i+z}\right| = 1$$

i.e., $|i - z| = |i + z|$

This is the locus of points z which are equidistant from i and $-i$, i.e., the line $Im(z) = 0$.

(w)



So the map $z \to w$ must take $Im(z) = 0$ onto $|w| = 1$ as required.

What about interior points? Pick one z-value $z = +i$ say $w(i) = 0$ i.e., $|w(i)| < 1$

Therefore upper $Im(z0 \geq 0 \longrightarrow |w| \leq 1$

In particular, consider $z = x + iy$ with $y = 0,\ x > 0$
$w(0) = 1,\ w(+\infty) = -1$
$w(1) = \dfrac{i-1}{i+1} = \dfrac{-(1-i)(1-i)}{(1+i)(1-i)} = -\dfrac{1}{2}(-2i) = i$
Now for $y = 0,\ x < 0$
$w(0) = 1,\ w(-\infty) = -1$
$w(-1) = \dfrac{i+1}{i-1} = -i$
Thus
PICTURE

Thus if $\phi_{(x,y)} = 1$ on OAB, $F_{(u,v)} = 1$ on $0'A'B'$
and if $\phi(x,y) = 0$ on OCD, $F(u,v) = 0$ on $0'C'D'$
Thus the boundary conditions on F are equivalent to solving the boundary conditions of ϕ of Q4A in (z).
Thus if we can solve $\nabla^2\phi = 0$ in z we map back to get back $\nabla^2 F = 0$ in w.
But we have solved $\nabla^2\phi = 0$ + boundary conditions in Q4A.
Hence

$$\phi(x,y) = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

If we can find $x(u,v),\ y(u,v)$ we have solved the problem <u>OR</u> we can say
$\phi(x,y) = Im\left(1 - \dfrac{1}{\pi}\log z\right)$
So

$$\begin{aligned} F(u,v) &= Im\left(1 - \frac{1}{\pi}\log z(w)\right) \\ &= Im\left(1 - \frac{1}{\pi}\log\left[i\left(\frac{1-w}{1+w}\right)\right]\right) \end{aligned}$$

<u>or</u>
if $z = i\dfrac{(1-w)}{(1+w)}$

a little algebra gives

$$x = \frac{2v}{(1+u)^2 + v^2}, \;\; y = \frac{1 - (u^2 + v^2)}{(1+u)^2 + v^2}$$

so $F(u,v) = \phi(x(u,v),\; y(u,v))$

$\Rightarrow \underline{F(u,v) = 1 - \frac{1}{\pi}\arctan\left[\frac{2v}{1-(u^2+v^2)}\right]}$