Question

Defining $\log(z-a) = \log|z-1| + i\arg(z-a)$, show that the real and imaginary parts of $\log(z-a)$ are harmonic functions in any region not containing $z = a$.

Answer

Method 1

If R does not contain a, then $w = \ln(z-a)$ is analytic in R. Hence the real and imaginary parts are harmonic in R.

Method 2

Let $z - a = re^{i\theta}$. Then $\log(z-a) = \log r + i\theta$.

Then in polars, Laplace's equation is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Can substitute $\phi = \log r$ or $\phi = \theta$ and they automatically satisfy this equation. Hence Re and Im parts are harmonic.